Automorphisms Fixing Subnormal Subgroups of Certain Infinite Soluble Groups

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DECLARATION

The work in this thesis is my own except where otherwise stated.

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ABSTRACT

The group of all automorphisms fixing setwise every subnormal subgroup, $Aut_{sn}(G)$, of a certain finitely generated infinite soluble group G with Max-n is studied. In particular, for a finitely generated infinite metabelian group G the complete structure of the group $Aut_{sn}(G)$ is obtained; and while for a finitely generated infinite Abelian-by-nilpotent group G some results are obtained. For the structure of the group $Aut_{sn}(G)$ crucial roles are played by $\omega(G)$, the Wielandt subgroup, and by the quotient group $\omega(G)/Z(G)$, where $Z(G)$ is the centre of G, and hence a sharper result on $\omega(G)$ of an infinite soluble group G with Max-n is obtained.

Furthermore, for a finitely generated infinite metabelian group G, we prove that the semidirect product $G \rtimes Aut_{sn}(G)$ is a finitely generated infinite metabelian group. Moreover similar results are also obtained for the class of finitely generated infinite Abelian-by-nilpotent groups as well.

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LIST OF NOTATION

Notation and terminology used throughout this thesis are mostly standard. Generally we follow Robinson [Rob82].

Elements and Groups

$A, B, \cdots, X, Z$	Sets, groups, rings, etc.
$a, b, \cdots, x, z$	Elements of a set.
x^y	$y^{-1}xy$, the conjugate of x by y.
$[x, y]$	$x^{-1}y^{-1}xy$, the commutator of x and y.
$[K, H]$	$\langle [k, h] : k \in K, h \in H \rangle$.
$[K_1, \cdots, K_n]$	$[[K_1, \cdots, K_{n-1}], K_n]$, for $n \geq 3$.
$[K, {}_mH]$	$[K, \underbrace{H, \cdots, H}_{m}]$.
$K \cong H$	K is isomorphic to H.
$N \leq H$	N is a subgroup of the group H.
$N < H$	N is a proper subgroup of the group H.
$N \lhd G$	N is a normal subgroup of a group G.
$H \, sn \, G$	H is a subnormal subgroup of G.
$C_G(H)$	centraliser of H in G.
$N_G(H)$	normaliser of H in G.
$K \rtimes H$	the semidirect product of K by H, with $K \lhd KH$ and $K \cap H = 1$.
$K \times H$	the direct product of K and H, with $K, H \lhd KH$ and $K \cap H = 1$.
$H \wr K$	the restricted wreath product of H and K, with the base group $B = H^{(K)}$ the direct product of isomorphic copies of H indexed by elements of K.
$Max\text{-}n$	the maximal condition on normal subgroups.
$Aut(G)$	the set of all automorphisms of a group G.

$Inn(G)$	the set of all inner automorphisms of a group G.
$PAut(G)$	the set of all power automorphisms of a group G.
$Aut_{sn}(G)$	the set of all automorphisms that fix every subnormal subgroup of a group G setwise.
$\|x\|$	the order of x (least positive integer m such that $x^m = 1$), or ∞.
$\|G\|$	the order of G.
$\|G : H\|$	the index of H in G (that is, $\|G\|/\|H\|$ for finite groups).
$exp(G)$	the exponent of G, the least common multiple of the orders of the elements of G.

Special Subgroups

$F(G)$	the Fitting subgroup of G.
$\omega(G)$	the Wielandt subgroup of G.
$Z(G)$	the centre of G.
$Z_n(G)$	the n^{th} term of the upper central series of G.
$\tau(G)$	the torsion radical of G, that is, the product of all the periodic normal subgroups of G.
$R(G)$	the finite radical of G, that is, the product of all the finite normal subgroups of G.
$G' = [G, G]$	the derived subgroup of a group G.
$\gamma_1(G) = G$	for $n = 1$.
$\gamma_n(G) = [\gamma_{n-1}(G), G]$	the n^{th} term of the lower central series of G, if $n \geq 2$.

Classes of Groups

$\mathfrak{X}$	denotes a class of groups.
$\mathfrak{G}$	denotes the class of all finitely generated groups.
$\mathfrak{N}$	denotes the class of all nilpotent groups.
$\mathfrak{A}$	denotes the class of all Abelian groups.
$\mathfrak{B}$	denotes the class of all polycyclic groups.
$\mathfrak{P}$	denotes the class of all periodic or torsion groups.
$\mathfrak{F}$	denotes the class of all finite groups.

$\mathfrak{F}_\pi$	denotes the class of all finite π-groups.
$\mathfrak{M}_n$	denotes the class of groups satisfying the maximal condition for normal subgroups (Max-n).
$\mathfrak{X}\mathfrak{Y}$	denotes the class of all $\mathfrak{X}$-by-$\mathfrak{Y}$ groups, i.e., extensions of an $\mathfrak{X}$ group by a $\mathfrak{Y}$ group.
$\mathbf{R}\mathfrak{F}$	denotes the class of all residually finite groups.

Group rings

$Ass(M)$	denotes the set of the primes associated to M, where M is an R-module, and R is a commutative ring.
$Ann(x)$	denotes the annihilator of x, that is, the ideal consisting of all elements $r \in R$ such that $rx = 0$, where $x \in M$, R is a commutative ring and M is an R-module.
$Ann(M)$	denotes the annihilator of M, that is, $\{x \in R : xM = 0\}$.
$\mathbb{Z}[G]$	denotes the integral group ring of G.
I_G	the augmentation ideal of $\mathbb{Z}[G]$, that is, $I_G = \langle g - 1 : 1 \neq g \in G \rangle$.

CHAPTER 1

1. INTRODUCTION

The study of the group $Aut_{sn}(G)$, the set of all automorphisms that fix setwise every subnormal subgroup of a group G, has grown out of the investigations of $PAut(G)$ and $Aut_n(G)$, the set of all automorphisms of a group G fixing setwise every subgroup (that is, power automorphisms), and every normal subgroup, respectively. Power automorphisms have been studied by many authors, mainly by Cooper [Coo68].

He showed that $PAut(G)$ is central in the automorphism group $Aut(G)$ of a group G, (Theorem 2.2.1, [Coo68]). As a consequence of this fundamental theorem, $PAut(G)$ fix the elements of the derived subgroup (Corollary 2.2.2, [Coo68]). Moreover power automorphisms fix elements of prime order (unless the group is Abelian, or has a Sylow subgroup which is an Abelian direct factor) and fix elements of infinite order (unless the group is Abelian)([Coo68], Theorem 5.1.1 and 4.2.2).

Almost 20 years after Cooper's work the study of the group $Aut_n(G)$ of a nilpotent group G was begun by Franciosi and de Giovanni [FdG87]. They showed that if G is a nilpotent p-group (with $p > 2$) of finite exponent, then $Aut_n(G)$ is an extension of a nilpotent p-group of finite exponent by a cyclic group of order dividing $p-1$; if G is a nilpotent p-group of infinite exponent or with $p = 2$, then $Aut_n(G)$ is nilpotent and its torsion subgroup is a p-group provided that G is non-Abelian; if G is nilpotent of class c and torsion free, then $Aut_n(G)$ is the extension of a torsion free nilpotent group of class $c-1$ by a cyclic group of order ≤ 2; (Theorems A and B, [FdG87]).

On the other hand, Robinson (Theorem 6, [Rob95]) proved that if H is a finite group, then there is a finite semisimple group G such that $Aut_n(G)/Inn(G)$ has a subgroup isomorphic with H, where $Inn(G)$ denotes the set of all inner automorphisms of a group G. If G is a finite soluble group, then $Aut_n(G)$ is soluble; indeed it is known that $Aut_n(G)$ is polycyclic if G is polycyclic ([FdG87]).

Note that $Aut_n(G)$ contains both $Inn(G)$ and $Aut_{sn}(G)$ ([Rob95]); and all of them are normal subgroups of $Aut(G)$.

The study of the group $Aut_{sn}(G)$ was also begun by Franciosi and de Giovanni in 1988. Since then many authors have also considered the structure of $Aut_{sn}(G)$ under various restrictions on the structure of a group G itself, in particular, when G is either finite or soluble.

For instance, when G is a soluble group, Franciosi and de Giovanni showed that the group $Aut_{sn}(G)$ is a metabelian group, and either finite or Abelian when G is a polycyclic group (Theorems A and B, [FdG88]).

Moreover, Dalle Molle ([DM95]) also proved that, when G is soluble, $Aut_{sn}(G)$ is a locally supersoluble group, provided that G is a Ĉernikov group, (a group which is an extension of a finite direct product of quasicyclic groups by a finite group), or its Fitting subgroup either is non-periodic or has finite exponent.

For a finite group G, Robinson (Theorem 1 and Corollary 3, [Rob95]) showed that the factor group

$$Aut_{sn}(G)/(Inn(G) \cap Aut_{sn}(G))$$

is soluble with derived length at most 4; and $Aut_{sn}(G)$ is soluble if and only if $\omega(G)$ is soluble.

Moreover, Cossey (Theorem 3 and Corollary 2, [Cos97]) proved that $Aut_{sn}(G)$ is metabelian and supersoluble when G is a finite group with $F^*(G) = F(G)$, where $F^*(G)$ denotes the generalised Fitting subgroup of G and $F(G)$ denotes the Fitting subgroup of G; and $Aut_{sn}(G)$ is soluble if and only if $F^*(G) = F(G)$ is soluble.

For a polycyclic group G, Cossey and Almazar (Theorem 1 and 2, [AC99]) showed that the factor group

$$Aut_{sn}(G)/(Inn(G) \cap Aut_{sn}(G))$$

is finite; and $Aut_{sn}(G)$ is infinite if and only if $\omega(G)/Z(G)$ is infinite, where the Wielandt subgroup $\omega(G)$ of G is defined to be the intersection of the normalisers of all subnormal subgroups of G, and $Z(G)$ denotes the centre of G.

In their paper [DF00](Corollary 4), Dardano and Franchi showed that the metabelian group $Aut_{sn}(G)$ acts by means of power automorphisms on its derived subgroup, when G is a nilpotent-by-finitely generated soluble group.

In the investigation of the group $Aut_{sn}(G)$ an important role is played by $\omega(G)$, the Wielandt subgroup of a group G. In general, for an arbitrary group G, the group $Aut_{sn}(G)$ acts trivially on the factor group $G/\omega(G)$, (Lemma 2.1, [FdG88]).

Most of the results obtained indicate that the structure of the group $Aut_{sn}(G)$ is controlled by $\omega(G)/Z(G)$. For instance when G is a finite group, $Aut_{sn}(G)$ is soluble if and only if $\omega(G)$ is soluble, (Corollary 3, [Rob95]); when G is polycyclic, $Aut_{sn}(G)$ is infinite if and only if $\omega(G)/Z(G)$ is infinite, (Theorem 1, [AC99]).

Moreover, the inner automorphisms induced by elements of $\omega(G)$ are contained in $Aut_{sn}(G)$, and so $Inn(G) \cap Aut_{sn}(G) \cong \omega(G)/Z(G)$.

Our main object in this thesis is to obtain more detailed information about the group $Aut_{sn}(G)$ of certain finitely generated **infinite** soluble groups with Max-n. In particular, for a finitely generated **infinite** metabelian group the detailed structure of the group $Aut_{sn}(G)$ is obtained, while for a finitely generated infinite Abelian-by-nilpotent group some results are obtained.

Critical for the structure of the group $Aut_{sn}(G)$ of an infinite soluble group G with Max-n are $\omega(G)$, the Wielandt subgroup, and the quotient group $\omega(G)/Z(G)$. For this purpose we need to investigate the Wielandt subgroup of an **infinite** soluble group with Max-n; this is done in Chapter 4.

Furthermore, we also look at whether the semidirect product of a finitely generated infinite metabelian group G by the group $Aut_{sn}(G)$ is a finitely generated infinite metabelian group, and affirmative answers are obtained. Moreover similar results are obtained for the class of finitely generated infinite Abelian-by-nilpotent groups.

We note that since the only non-trivial power automorphism of infinite Abelian groups is inversion we only consider **infinite non-Abelian groups** in this thesis.

The overall structure of this thesis is as follows. In Chapter 2 we present a more detailed setting for the work in this thesis, give a collection of results on the group of automorphisms fixing subnormal subgroups of a group so far while also discussing some preliminary results.

Chapter 3 collects some known results from commutative algebra which will be used in Chapter 5, and also discusses some preliminary results.

In chapter 4 we give results on the Wielandt subgroup of an infinite soluble group G with Max-n. This chapter contains the following main results.

Theorem A *Let G be an infinite soluble group with Max-n. Then the Wielandt subgroup $\omega(G)$ of G is an Abelian group.*

This result yields the following Corollary.

Corollary B *Let G be a finitely generated infinite Abelian-by-nilpotent group.*

(i) $\omega(G)$ centralises $F(G)$, that is $[F(G), \omega(G)] = 1$.

(ii) If B is an Abelian normal subgroup of G, then $B\omega(G)$ is an Abelian group.

The next main result tells us that $C_G(\omega(G))$ has finite index in G.

Theorem C *Let G be a finitely generated infinite Abelian-by-nilpotent group. Then $C_G(\omega(G))$ has finite index in G.*

In fact from Theorem C since $\omega(G)$ is central in $C_G(\omega(G))$ and $C_G(\omega(G))$ satisfies Max-n and so is finitely generated, it follows that $\omega(G)$ is finitely generated.

The next result is about the action of the group $Aut_{sn}(G)$ on $\omega(G)$.

Theorem D *Let G be a finitely generated Abelian-by-nilpotent group and set $\Gamma = Aut_{sn}(G)$.*

(i) If G' is finite, then Γ has a subgroup of index at most two centralising $F(G)$.

(ii) If G' is infinite, then Γ centralises $F(G)$, and hence $[\Gamma, \omega(G)] = 1$.

Chapter 5 presents the main results of this work on the group $Aut_{sn}(G)$ of a finitely generated infinite metabelian group G and their proofs.

The main results of this chapter are the following.

Theorem E *Let G be a finitely generated metabelian group.*

(a) If G' is infinite, then

(i) The group $Aut_{sn}(G)$ is a finite Abelian group;

(ii) $[G, Aut_{sn}(G)] \leq R(G)$; and

(iii) $Z(G)$ has finite index in $\omega(G)$.

(b) If G' is finite, then the group $Aut_{sn}(G)$ is a finite metabelian group.

Note that if G' is finite, then G is a centre-by-finite group, and so $Z(G)$ has finite index in G, and hence $Z(G)$ has finite index in $\omega(G)$.

The proof of Theorem E divides into two parts, when $\tau(G) = 1$, and when $\tau(G) \neq 1$, where $\tau(G)$ is the torsion radical of G.

The next result gives a proof of Theorem E, for the trivial torsion radical case. The methods of the proof of this critical result, Lemma F, depend heavily upon the techniques and results from commutative algebra.

Lemma F *Let G be a finitely generated infinite metabelian group with $\tau(G) = 1$. Then there are no non-trivial automorphisms that fix every subnormal subgroup of G setwise, that is $Aut_{sn}(G) = 1$, and $\omega(G) = Z(G)$.*

The next result is a special case of Theorem E, and we prove this result as an intermediate step in the proof of Theorem E. The methods of the proof depend upon group theoretic techniques alone.

Lemma G *Let G be a finitely generated infinite metabelian group with $\tau(G) \neq 1$. If $R(G) = 1$, then $Aut_{sn}(G) = 1$, and $\omega(G) = Z(G)$.*

Finally in this chapter we prove the following Corollary to Theorem E.

Corollary H *Let G be a finitely generated metabelian group. Assume that G', the derived subgroup of G, is torsion free as $\mathbb{Z}[G/G']$-module. Then the group $Aut_{sn}(G) = 1$, and $\omega(G) = Z(G) = 1$.*

Chapter 6 is primarily concerned with the question of whether the semidirect product of a finitely generated infinite metabelian group G by the group $Aut_{sn}(G)$ is a finitely generated infinite metabelian group. The main results of this chapter are the following.

Theorem I *Let G be a finitely generated infinite metabelian group and set $\Gamma = Aut_{sn}(G)$. Let $\Phi = G \rtimes \Gamma$, the semidirect product of G by Γ. Then Φ is a finitely generated metabelian group, and $\omega(\Phi)$ is an Abelian group.*

The following Corollary to Theorem I gives more detailed information about the structure of Φ.

Corollary J *Let G be a finitely generated infinite metabelian group and set $\Gamma = Aut_{sn}(G)$. Let $\Phi = G \rtimes \Gamma$, the semidirect product of G by Γ. Then* *(i)* $\omega(\Phi) \leq \omega(G)\Gamma$;

(ii) If G' is infinite, then $F(\Phi) = F(G)\Gamma$;

(iii) If G' is finite, then $F(\Phi) \leq F(G)\Gamma$; *and*

(iv) $R(\Phi) = R(G)\Gamma$.

We give examples which show that $Aut_{sn}(G)$ can be non-Abelian, need not centralise the Wielandt subgroup and $Aut_{sn}(G)$ can contain outer automorphisms.

Chapter 7 presents some results on the group $Aut_{sn}(G)$ of a finitely generated infinite Abelian-by-nilpotent group G. The main results of this chapter are the following.

Theorem K *Let G be a finitely generated Abelian-by-nilpotent group.*

(i) If G' is infinite, then the group $Aut_{sn}(G)$ is a finitely generated Abelian group.

(ii) If G' is finite, then the group $Aut_{sn}(G)$ is a finite metabelian group.

This result easily gives the following Corollaries.

Corollary L *Let G be a finitely generated Abelian-by-nilpotent group with G' infinite and set $\Gamma = Aut_{sn}(G)$.*

If $\omega(G) = Z(G)$ then Γ is a finite Abelian group, and

$$[G, \Gamma] \leq R(G).$$

In particular, if $R(G) = 1$, then $\Gamma = 1$.

Corollary M *Let G be a finitely generated Abelian-by-nilpotent group and let $H = C_G(\omega(G))$, and set $\Gamma = Aut_{sn}(G)$.*

If H' is infinite, then Γ is a finite Abelian group, and

$$[\Gamma, G] \leq \omega(G) \leq Z(H) \leq R(G).$$

In particular, if $R(G) = 1$, then $\Gamma = 1$.

For a finitely generated metabelian group G, we proved that the semidirect product $G \rtimes \Gamma$ is a finitely generated infinite metabelian group, and $\omega(G \rtimes \Gamma)$ is an Abelian group, (see Theorem I), where $\Gamma = Aut_{sn}(G)$.

We also give similar results which generalise the metabelian case to the class of finitely generated Abelian-by-nilpotent groups.

Theorem N *Let G be a finitely generated Abelian-by-nilpotent group and $\Gamma = Aut_{sn}(G) = \Psi \rtimes \langle \delta \rangle$, where*

$$\Psi = C_\Gamma(G/G') = \{\alpha \in \Gamma : g^\alpha G' = gG', \ for\ all\ gG' \in G/G'\}$$

and $\delta \in \Gamma \setminus \Psi$. Let A be a maximal Abelian normal subgroup of G such that G/A is nilpotent, and set $B = A\Psi$.

(i) The semidirect product of G by Γ, $\Phi = G \rtimes \Gamma$, is a finitely generated Abelian-by-nilpotent group.

(ii) Φ/B and G/A have the same nilpotency class.

(iii) $\omega(\Phi)$ is an Abelian group.

This result gives the following Corollary (compare to Corollary J).

Corollary O *Let G be a finitely generated Abelian-by-nilpotent group and let $\Phi = G \rtimes \Gamma$, the semidirect product of G by Γ.*

(i) $\omega(\Phi) \leq \omega(\Phi)\Gamma$;

(ii) If G' is infinite, $F(\Phi) = F(G)\Gamma$; and

(iii) If G' is finite $F(\Phi) \leq F(G)\Gamma$; and

(iv) If Γ is finite, then $R(\Phi) = R(G)\Gamma$.

We conclude by asking one of the key question raised by this thesis. In general, we ask the following question.

> *How far is the Wielandt subgroup of an infinite soluble group with Max-n from the centre?*

For a finitely generated infinite metabelian group G the question was settled by Theorem E, that is $Z(G)$ has finite index in $\omega(G)$.

Here we also ask similar sort of question for the class of finitely generated Abelian-by-nilpotent groups.

Question P *Let G be a finitely generated Abelian-by-nilpotent group. Is $\omega(G)/Z(G)$ is finite?*

CHAPTER 2

2. PRELIMINARIES

This chapter is devoted primarily to reminding the reader of the basic definitions and known results that will be used in the remaining chapters, and some preliminary results will also be presented. In section 1 we give definition of subnormal subgroups. Section 2 presents the well known results of Hall in properties of finitely generated soluble groups. Section 3 introduces the product of all the periodic normal subgroups (torsion radical) of a group, and the finite radical of a group. Section 4 discusses special nilpotent normal subgroups. In section 5 we give the basic definition of the Wielandt subgroup of a group. Section 6 presents residually nilpotent groups. Section 7 discusses stability groups. In section 8 we give a collection of results on the group of automorphisms fixing subnormal subgroups of a group so far while also discussing the role of the Wielandt subgroup.

Concepts and ideas that are undefined or not explained in this work can be found in the books [Rob82] of Robinson for group theory and [Bou72] of Bourbaki for commutative algebra.

Notation used in this work is mostly standard and based on that appearing in articles [Hal59] of Hall, [Seg74] of Segal, and the book [Rob82] of Robinson. The reader is referred to the list of notations on pages $iii - iv$.

Every result which is cited in this work appears exactly as it does in its source, and some of the notation will differ from that used in the body of this thesis.

Throughout this thesis all groups are assumed to be infinite, unless otherwise specified.

2.1 Subnormal subgroups

As subnormal subgroups play an important role in our study we remind the reader of basic definitions and properties of subnormal subgroups of a group that we will use.

Definition 2.1.1 *A subgroup H of a group G is subnormal in G, denoted by $H \,\mathrm{sn}\, G$, if there is a finite chain of subgroups*

$$H = H_0 \lhd H_1 \lhd, \cdots, \lhd H_{n-1} \lhd H_n = G$$

stretching from H to G, each member in the chain being normal in the next.

Recall that if G is nilpotent then every subgroup of G is subnormal, (see [Rob72] part 1, p.49).

The following lemma will be used.

Lemma 2.1.2 (Proposition 1.1.2, [LS87]) *Let $H \,\mathrm{sn}\, G$ and K be a subgroup of G. Then $H \cap K \,\mathrm{sn}\, K$. In particular $H \,\mathrm{sn}\, L$ whenever L is a subgroup of G containing H.*

2.2 Finitely generated soluble groups

This section contains the properties of finitely generated soluble groups that satisfy Max-n that we will need.

Recall that a group G satisfies *Max-n* if and only if it satisfies the maximal condition for normal subgroups, and hence $G \in \mathfrak{M}_n$.

Definition 2.2.1 *A group G is a residually-$\mathfrak{X}$ group if*

$$\cap\{N : N \lhd G,\, G/N \in \mathfrak{X}\} = 1,$$

where $\mathfrak{X}$ is a class of groups, and **R**$\mathfrak{X}$ *denotes the class of residually-$\mathfrak{X}$ groups.*

If $K \lhd H, K \lhd G$ and $H \lhd G$, then we call H/K is a *chief factor* of G if and only if it is a minimal normal subgroup of G/K.

Hall summarised the properties of finitely generated soluble groups in the following Theorem, which will be used throughout this thesis without further comment.

Theorem 2.2.2 (Theorem 5, [Hal61]) *Let G be a group.*

(i) If $G \in \mathfrak{G} \cap \mathfrak{A}\mathfrak{N}$, then

(a) G satisfies Max-n; and

(b) *G is residually finite.*

(ii) *If $G \in \mathfrak{G} \cap \mathfrak{M}\mathfrak{N}$, then*

(c) *all chief factors of G are finite; and*

(d) *all maximal subgroups of G are of finite index.*

It is convenient to use abbreviations for certain properties of groups. Those often required are in the following subsection.

2.2.1 Poly properties of groups

Recall that a group G satisfies *Max* if and only if it satisfies the maximal condition for subgroups.

Max : the maximal condition for subgroups.
FG : the property of being finitely generated.
Max-n : the maximal condition for normal subgroups.
FR : the property of being finitely related.

We say that a property $\mathcal{P}$ of groups is a *poly property* if, whenever N and G/N have the property $\mathcal{P}$, so has G, where N is a normal subgroup of G.

The result we need on poly properties of groups is due to Hall [Hal54].

Lemma 2.2.3 (Lemma 1, [Hal54]) *FG, Max, Max-n, and FR are poly properties.*

Let $\mathcal{P}$ be any property of groups, and let $\mathcal{Q}$ be another property of groups. A group G is a $\mathcal{P}$-by-$\mathcal{Q}$ if G has normal subgroup N such that N has property $\mathcal{P}$ and G/N has property $\mathcal{Q}$.

2.3 Periodic normal subgroups

In this section we show that a group G with Max-n has a unique maximal periodic normal subgroup which contains the unique maximal finite normal subgroup of G.

Recall that an element of finite order is called a *periodic* element, sometimes called a *torsion* element. If all elements of a group are periodic (or torsion), the group is called *periodic* (or *torsion*). If at least one element of a group is of infinite order, the group is called *non-periodic*. The other extreme case is that where all the elements (except the identity element) have infinite order and we then call the group *torsion free* .

Recall also that the *exponent* of a group G, $exp(G)$, is the least common multiple of all the orders of the elements of G. If the orders of the elements of a group G are finite and bounded, the group is said to have *finite exponent.*

Proposition 2.3.1 *Let G be a group. Let $N_1, \cdots, N_k$ be periodic normal subgroups of a group G, and let $L_1, \cdots, L_n$ be finite normal subgroups of a group G, where k and n are positive integers.*

(i) Then the subgroup $\langle N_1, \cdots, N_k\rangle = N_1N_2\cdots N_k \leq G$ is a periodic normal subgroup of G; and

(ii) The subgroup $\langle L_1, \cdots, L_n\rangle = L_1L_2\cdots L_n \leq G$ is a finite normal subgroup of G.

Proof. (i) An obvious induction reduces to the case $k = 2$. Set $N = N_1$, and $M = N_2$.

(a) Suppose that $N \cap M = 1$. Let $n = exp(N)$, and $m = exp(M)$. If $x \in N$ and $y \in M$, then $x^n = 1$ and $y^m = 1$. Since x and y commute, that is, $[N, M] = 1$, then $(xy)^{nm} = x^{nm}y^{nm} = (x^n)^m(y^m)^n = 1$. Hence xy is periodic element of NM and so NM is a periodic normal subgroup of G.

(b) Suppose that $N \cap M \neq 1$. Put $L = N \cap M$, then L is clearly a periodic normal subgroup of G. And also N/L and M/L are periodic normal subgroups of G/L, since N and M are periodic normal subgroups of G. Put $\bar{N} = N/L$, $\bar{M} = M/L$, $\bar{G} = G/L$, and $\bar{1} = L/L$. Clearly $\bar{N} \cap \bar{M} = \bar{1}$. Then by (a) above $\bar{N}\bar{M}$ is a periodic normal subgroup of $\bar{G}$. It follows easily that NM is a periodic normal subgroup of G. The result follows.

The proof of (ii) is trivial. ∎

Recall that a class of groups is said to be *radical* if and only if it is closed with respect to forming normal products, homomorphic images and extensions, (Theorem 1.32, [Rob72] part 1). If $\mathfrak{X}$ is a class of groups and G is any group, the $\mathfrak{X}$-*radical* of G is the product of all the normal $\mathfrak{X}$-subgroups of G. It is well known that the class of periodic groups $\mathfrak{P}$ is closed with respect to forming normal products, homomorphic images and extensions, and so a *radical* class, (section 1.4D, [Rob72] part 1).

If G is a group, denote by $\tau(G)$ the group generated by all the periodic normal subgroups of G. So by Proposition 2.3.1(i), $\tau(G)$ is periodic, and so $\tau(G)$ is the $\mathfrak{P}$-*radical* of G; and it is also characterstic in G.

Since $\tau(G)$ is the $\mathfrak{P}$-*radical* of G, we will follow Robinson [Rob72] and call $\tau(G)$ the torsion radical of G.

Definition 2.3.2 *Let G be a finitely generated metabelian group. The torsion radical of G is*

$$\tau(G) = \langle N : N \lhd G \text{ and } N \text{ is periodic}\rangle.$$

Lemma 2.3.3 *Let G a group and $G \in \mathfrak{M}_n$. Then there is a unique maximal finite normal subgroup $R(G)$ of G.*

Proof. If $R(G) = 1$, then there is nothing to prove. Since G satisfies Max-n, there exists a maximal finite normal subgroup of G. If H and K are two maximal finite normal subgroups of G, then, by Proposition 2.3.1(ii), HK is a finite normal subgroup of G. Hence $H = HK = K$. Therefore there exists a unique maximal finite normal subgroup $R(G)$ of G. ∎

We call $R(G)$ the finite radical of G.

Since a finitely generated metabelian group G satisfies Max-n, by Theorem 2.2.2, there exists a unique torsion radical $\tau(G)$ of G, and by Lemma 2.3.3, a unique finite radical $R(G)$ of G, and clearly $\tau(G)$ always contains $R(G)$. $\tau(G)$ and $R(G)$ will be used throughout this work without further comment.

Lemma 2.3.4 *Let G be a group with $\tau(G) \neq 1$. Then $\tau(G/\tau(G)) = 1$.*

Proof. Suppose, on the contrary, that $\tau(G/\tau(G)) \neq 1$. Now let

$$\tau(G/\tau(G)) = H/\tau(G).$$

If $n = exp(H/\tau(G))$, then $h^n \in \tau(G)$, for all $h \in H$. But since $\tau(G)$ is a periodic group, there is a positive integer m such that

$$(h^n)^m = h^{nm} = 1,$$

and so that $H \leq \tau(G)$ and $H/\tau(G) = 1$. This is a contradiction, since

$$\tau(G/\tau(G)) \neq 1,$$

by our supposition. The result follows. ■

2.4 Nilpotent normal subgroups

Since we will often make use of the Fitting subgroup of a group, will give the basic definition of this subgroup and its properties that will be used. We will also give elementary facts about maximal Abelian normal subgroups.

Accordingly, we define here, ([Rob82], p.129).

Definition 2.4.1 *The Fitting subgroup $F(G)$ of a group G is the subgroup generated by all the normal nilpotent subgroups of G, that is*

$$F(G) = \langle N : N \lhd G \textit{ and } N \in \mathfrak{N} \rangle.$$

Recall that in any group the product of two normal nilpotent subgroups is itself nilpotent, (Fitting's theorem 5.2.8, [Rob82]).

Observe that $F(G)$ is nilpotent if and only if G has a maximal normal nilpotent subgroup L, in which case $F(G) = L$.

Note that if $G \in \mathfrak{M}_n$, $F(G)$ is the unique maximal normal nilpotent subgroup of G, as G satisfies Max-n.

The following elementary lemma will be used frequently.

Lemma 2.4.2 *Let G be a group and $G \in \mathfrak{M}_n$. If K is a subnormal subgroup of G, then $F(K) = F(G) \cap K$.*

Proof. Let $G \in \mathfrak{M}_n$. Since G satisfies Max-n, $F(G)$ is the unique maximal normal nilpotent subgroup of G, and so

$$F(K) \leq F(G) \cap K.$$

On the other hand, $F(G) \cap K$ is a normal nilpotent subgroup of K, and so

$$F(G) \cap K \leq F(K).$$

Now it follows that $F(K) = F(G) \cap K$. Hence the result follows. ■

Next we will give elementary facts about maximal Abelian normal subgroups which are used frequently.

Lemma 2.4.3 *Let G be a group. If A is a maximal Abelian normal subgroup of G, then $C_G(A) = A$.*

Proof. Clearly $A \leq C_G(A)$. Put $C = C_G(A)$. Let $1 \neq x \in C$ and $x \notin A$. But $[x, A] = 1$, and so $\langle x, A\rangle$ is Abelian and normal in G. Hence by maximality of A, $\langle x, A\rangle = A$, and so $x \in A$. Therefore the result follows. ■

The next Corollary is a special case of Lemma 2.4.3 which we will often use.

Corollary 2.4.4 *Let G be a finitely generated group. If A is a maximal Abelian normal subgroup of G containing G', then $C_G(A) = A$.*

2.5 The Wielandt subgroup

The Wielandt subgroup plays an important role in the investigation of the group of automorphisms fixing subnormal subgroups, and so in this section we remind the reader of its definition, and its some elementary properties.

In Chapter 4 we will give more discussion of and results on the Wielandt subgroup of an infinite soluble group with Max-n.

Definition 2.5.1 *The Wielandt subgroup $\omega(G)$ of a group G is defined to be the intersection of the normalisers of all subnormal subgroups of G, that is,*

$$\omega(\mathrm{G}) = \bigcap_{\mathrm{H\,sn\,G}} \mathrm{N_G(H)}.$$

It is clear from the definition that the Wielandt subgroup $\omega(G)$ of a group G is always a T-group (a group in which normality is a transitive relation) and always contains the centre $Z(G)$ of G. If G is a soluble group, then $\omega(G)$ is a metabelian T-group, as every soluble T-group is metabelian by Theorem 2.3.1 of Robinson in [Rob64].

We often use the following elementary Lemma throughout this thesis without further comment.

Lemma 2.5.2 *Let G be a group and N be a normal subgroup of G. Then $\omega(G)N/N \leq \omega(G/N)$.*

Proof. Let $uN \in \omega(G)N/N$, where $u \in \omega(G)$, and let H/N be a subnormal subgroup of G/N. Then H sn G, by the Correspondence Theorem, and so $(H/N)^{uN} = H/N$, which implies that $uN \in \omega(G/N)$, and hence

$$\omega(G)N/N \leq \omega(G/N).$$

■

Next we remind the reader of the following terminologies *weak subgroup*, *strong subgroup* and *power automorphism*.

Recall that if G is a non-periodic group, the subgroup of G generated by the elements of infinite order is called the *weak* subgroup of G and is denoted by $W(G)$. A non-periodic group G is called *weak* if $W(G) = G$; otherwise it is called *strong*. If the subgroup $T(G)$ of a group G, generated by the elements of finite order, is a proper subgroup of G, then G is *weak* since it is generated by the elements outside of $T(G)$. It follows therefore that non-periodic locally nilpotent groups are *weak*, (section 4.1, [Coo68]).

Recall also that a *power automorphism* of a group G is an automorphism mapping every subgroup of G on itself. It is well known that $PAut(G)$, the set of all power automorphisms of G, is an Abelian normal subgroup of $Aut(G)$, the set of all automorphisms of G, (Theorem 2.1.1, [Coo68]).

Now the following result due to Cooper [Coo68] will be used.

Lemma 2.5.3 (Corollary 4.2.3, [Coo68]) *Let G be a weak group.*

(i) If G is Abelian, then $|PAut(G)| = 2$.

(ii) If G is non-Abelian, then $|PAut(G)| = 1$.

2.6 Residually nilpotent groups

Since we often use the properties of residually nilpotent groups in our study, we will give some known results which will be used frequently.

Lemma 2.6.1 *Let G be a group. Let N and M be normal subgroups of G. If N is nilpotent and M is residually nilpotent, then NM is residually nilpotent.*

Proof. Since $M \in \mathbf{R}\mathfrak{N}$, given $L \in \{L : L \lhd M, M/L \in \mathfrak{N}\}$ and

$$\mathfrak{L} = L \cap \{L : L \lhd M, M/L \in \mathfrak{N}\} = 1.$$

If $L \in \mathfrak{L}$, then $NL/L \cong N/L \cap N$ is a nilpotent group, and so M/L and NL/L are nilpotent normal subgroups of G/L. Then their product

$$(NL/L).(M/L) = NM/L$$

is also nilpotent by Fitting's Theorem 5.2.8 in [Rob82]. Hence it follows easily that NM is residually nilpotent. ■

The following important lemma is due to Cossey [Cos91], (Lemma (i)), and is a generalization of Schenkman's Theorem in [Sch60]. It will be used throughout this thesis, and so we shall include the proof of (i), and refer the reader to the proof of (ii) in [Cos91].

Lemma 2.6.2 (Lemma, [Cos91]) *Let G be a residually nilpotent group.*

(i) The Wielandt subgroup $\omega(G)$ of G is contained in $Z_2(G)$, the second centre of G.

(ii) If G is a residually finite p-group for some prime p and $\omega(G)$ is torsion free, then $\omega(G) = Z(G)$.

Proof. (i) Let G be a residually nilpotent group. Then by definition 2.2.1, given $1 \neq g \in G$, there exists $N_g \lhd G$ such that $g \notin N_g$ and G/N_g is nilpotent, moreover, $\bigcap_{1\neq g\in G} N_g = \{1\}$. Let $\omega(G)$ be the Wielandt subgroup of G.

Observe that $\omega(G)N_g/N_g \leq \omega(G/N_g)$, by Lemma 2.5.2.

Since G/N_g is nilpotent, by Schenkman's Theorem in [Sch60],

$$\omega(G/N_g) \leq Z_2(G/N_g).$$

Let $x, y \in G$, and $w \in \omega(G)$. Now since $[G/N_g, Z_2(G/N_g)] \leq Z(G/N_g)$, then $[wN_g, xN_g] = [w, x]N_g \in Z(G/N_g)$, and so $[wN_g, xN_g, yN_g] = [w, x, y]N_g = N_g$. Hence $[w, x, y] \in N_g$, for all $1 \neq g \in G$. It follows that $[w, x, y] = 1$, as $\cap_{1\neq g\in G} N_g = \{1\}$. Now $[w, x]^y = [w, x]$, for all $y \in G$, which gives that

$$[w, x] \in Z(G),$$

and so $w \in Z_2(G)$. Therefore $\omega(G) \leq Z_2(G)$. ■

In his classification of properties of finitely generated soluble groups, (Theorem 2.2.2(i)(b)), Hall showed that every finitely generated Abelian-by-nilpotent group is residually finite. In more detail, Segal proved that every finitely generated Abelian-by-nilpotent has a normal subgroup of finite index which is residually finite-nilpotent, (see Theorem 2.6.3); and he also extended this result to finitely generated Abelian-by-polycyclic groups, (see Theorem A and Corollary B of [Seg75]).

The next result due to Segal ([Seg74] will be used throughout this thesis.

Theorem 2.6.3 (Corollary 1, [Seg74]) *Let $G \in \mathfrak{G} \cap \mathfrak{A}\mathfrak{N}\mathfrak{F}$, and let M be an Abelian normal subgroup of G with $G/M \in \mathfrak{N}\mathfrak{F}$. Then there exists a finite set σ of primes such that*

$$G \in (R(\mathfrak{F}_{\pi(M)\cup\{p\}} \cap \mathfrak{N}))\mathfrak{F}$$

for every prime $p \notin \sigma$; where $\pi(M)$ denotes the set of primes q such that M contains an element of order q.

2.7 Stability groups

This section reminds us of the definition of a stability group and the main result due to Hall about such groups.

Recall that if G is a nilpotent group, then its *nilpotency class* is the length of the upper central series or the length of the lower central series, and so G is nilpotent of class $\leqq c$ if and only if $Z_c(G) = G$ if and only if $\gamma_{c+1}(G) = 1$, (5.1.9, [Rob82]).

Definition 2.7.1 *Let*

$$1 = G_m \lhd G_{m-1} \lhd, \cdots, \lhd G_1 \lhd G_0 = G$$

be a chain (or series) of subgroups of the group G. We define the stability group of this chain (or series) to be the group $Aut(G)$ of all automorphisms α of G such that

$$(G_i x)^{\alpha} = G_i x$$

for all $x \in G_i$ for each $i = 1, 2, \cdots, m$. In other words, all the factors G_{i-1}/G_i are fixed elementwise by $Aut(G)$, and each G_i invariant under $Aut(G)$, for each $i = 1, 2, \cdots, m$. Then $Aut(G)$ is said to be stabilise the series.

The main result we need on *stability groups* is due to Hall [Hal58].

Theorem 2.7.2 (Theorem 1, [Hal58]) *Let G be a group. Suppose that $Aut(G)$ stabilises a series*

$$1 = G_m \lhd G_{m-1} \lhd, \cdots, \lhd G_1 \lhd G_0 = G.$$

Then $Aut(G)$ is nilpotent of class at most $\frac{1}{2}m(m-1)$.

2.8 Automorphisms fixing subnormal subgroups

This section features some results on the group of all automorphisms fixing subnormal subgroups of a group setwise, some of which will be frequently used throughout this thesis.

We denote by $Aut_{sn}(G)$ the set of all automorphisms that fix every subnormal subgroup of a group G setwise, that is

$$Aut_{sn}(G) = \{\alpha \in Aut(G) : H^{\alpha} \leq H, \ for \ all \ H \ sn \ G\}.$$

The group $Aut_{sn}(G)$ have been studied by many authors since its introduction by Franciosi and de Giovanni in [FdG88], and more recently in Dalle Molle [DM95], Robinson [Rob95], Cossey [Cos97], Cossey and Almazar [AC99], and Dardano and Franchi [DF00].

In the next subsections we will list some of these known results.

2.8.1 Soluble groups

In this subsection we give known results on the group $Aut_{sn}(G)$ of a soluble group G.

Franciosi and de Giovanni proved the following main results.

Theorem 2.8.1 (Theorem A and B, [FdG88]) *Let G be a group.*

(a) If G is soluble, then the group $Aut_{sn}(G)$ is metabelian.

(b) If G is polycyclic, then the group $Aut_{sn}(G)$ is either finite or Abelian.

Moreover, Dalle Molle also showed that:

Theorem 2.8.2 (Dalle Molle, [DM95]) *Let G be a soluble group. If G is a Černikov group, or its Fitting subgroup either is non-periodic or has finite exponent, then the group $Aut_{sn}(G)$ is a locally supersoluble group.*

Cossey and Almazar further extended these results for polycyclic case, and they proved a necessary and sufficient condition for $Aut_{sn}(G)$ to be infinite. The results we need from [AC99] are the following.

Theorem 2.8.3 (Theorem 1 and 2, [AC99]) *Let G be a polycyclic group.*

(a) The group $Aut_{sn}(G)$ is infinite if and only if $\omega(G)/Z(G)$ is infinite; and

(b) The group $Aut_{sn}(G) \cap Inn(G)$ has finite index in $Aut_{sn}(G)$.

Dardano and Franchi in [DF00] also proved the following result.

Theorem 2.8.4 (Corollary 4, [DF00]) *If G is a nilpotent-by-finitely generated soluble group, then the metabelian group $Aut_{sn}(G)$ acts by means of power automorphisms on its derived subgroup.*

2.8.2 Finite groups

In this subsection we give known results on $Aut_{sn}(G)$ when G is a finite group.

Throughout this subsection let G be an arbitrary finite group. Then the structure of the group $Aut_{sn}(G)$ is very restricted.

One of the main results is the following.

Theorem 2.8.5 (Theorem 1, and Corollary 3, [Rob95]) *Let G be a finite group and write $\Gamma = Aut_{sn}(G)$. Then*

(i) $\Gamma/\Gamma \cap Inn(G)$ is soluble with derived length at most 4; moreover the derived length is at most 3 if $H^1(G/S, Z(S)) = 0$, where S is the soluble radical of G.

(ii) The group $Aut_{sn}(G)$ is soluble if and only if $\omega(G)$ is soluble.

In his paper [Cos97], Cossey gave more detailed information about the structure of $Aut_{sn}(G)$ by focussing on its action on $F^*(G)$, the generalised Fitting subgroup of G, and $F(G)$, the Fitting subgroup of G . His main results are as follows.

Theorem 2.8.6 (Theorem 1 and 3, and Corollary 2, [Cos97]) *Let G be a finite group and put $\Gamma = Aut_{sn}(G)$.*

i) Then we have $E(\Gamma) = E(G)^{\tau}$ and $\Gamma/(E(\Gamma)C)$ soluble with derived length at most 3, where $E(\Gamma)$ is the layer of Γ, τ is the natural homomorphism of G onto $Inn(G)$, and $C = C_{\Gamma}(F^*(G))$.

ii) If $F^*(G) = F(G)$, then Γ is metabelian and supersoluble. Moreover, if π is the set of primes dividing $|F(G)|$, then $|\Gamma/F(\Gamma)| \leq \prod_{p\in\pi}(p-1)$ and the nilpotency class of $F(\Gamma)$ is bounded by $max_{p\in\pi}\{e_p\}$, where p^{e_p} is the exponent of the Sylow p-subgroup of $F(G) \cap \omega(G)$.

iii) The group Γ is soluble if and only if $F^*(G) = F(G)$.

One can see that from these results the structure of the group $Aut_{sn}(G)$ is very restricted when G is a finite group.

2.8.3 The role of the Wielandt subgroup

In this subsection we discuss the role of the Wielandt subgroup $\omega(G)$ of a group G in the investigation of the group $Aut_{sn}(G)$ from most of the above known results.

Franciosi and de Giovanni in their paper [FdG88] proved the following important lemma for any arbitrary group.

Lemma 2.8.7 (Lemma 2.1, [FdG88]) *Let G be any group. Then the group $Aut_{sn}(G)$ acts trivially on the factor group $G/\omega(G)$.*

In the study of the group $Aut_{sn}(G)$ a major role is played by the Wielandt subgroup $\omega(G)$ of a group G as we can see from the above results.

For instance, Theorem 2.8.5(ii) of Robinson tell us, the group $Aut_{sn}(G)$ is soluble if and only if $\omega(G)$ is soluble, when G is a finite group.

When G is a polycyclic group, Theorem 2.8.3(a) of Cossey and Almazar, gives us more precise structure, that is, the group $Aut_{sn}(G)$ is infinite if and only if $\omega(G)/Z(G)$ is infinite.

Most of the results obtained indicate that the structure of the group $Aut_{sn}(G)$ is controlled by $\omega(G)/Z(G)$. The inner automorphisms induced by elements of

$\omega(G)$ are precisely contained in $Aut_{sn}(G)$ and hence $Aut_{sn}(G) \cap Inn(G)$ is isomorphic to $\omega(G)/Z(G)$.

Critical for the structure of the group $Aut_{sn}(G)$ of a group G are $\omega(G)$, the Wielandt subgroup, and the quotient group $\omega(G)/Z(G)$. For this purpose we need to study $\omega(G)$ in detail. We postpone the study of this subgroup for infinite soluble groups with Max-n to Chapter 4.

CHAPTER 3

3. RELATED COMMUTATIVE ALGEBRA

This chapter collects known results from commutative algebra which will be used in Chapter 5. In section 1 we discuss the connection of commutative algebra with finitely generated metabelian groups. In section 2 we give the basic definition of the associated set of primes of a module. Section 3 contains the primary decomposition of a module and some known results, while section 4 gives the basic definition of the augmentation ideal. In section 5 we give some technical results.

3.1 Connection with commutative algebra

In the 1950s the publication of P. Hall's three fundamental papers [Hal54], [Hal59], and [Hal61], first drew attention to the importance of commutative algebra in the theory of finitely generated soluble groups. For example, it follows from Hall's work that a finitely generated metabelian group satisfies *Max-n*, the maximal condition on normal subgroups, is residually finite, has all chief factors finite, and has all maximal subgroups of finite index.

Let B be an Abelian normal subgroup of a finitely generated metabelian group G for which G/B is Abelian, so that $G' \leq B$. Put $H = G/B$. For each $g \in G$, $b \in B$, and $h = gB \in H$, we may define $b^h = g^{-1}bg = b^g$, (that is, G acts on B by conjugation), which is independent of the choice of g in the coset h. Let $R = \mathbb{Z}[H]$ be the integral group ring of H. If $r \in R$ we can write $r = n_1h_1 + ... + n_kh_k$, where n_i are integers and $h_i = g_iB \in H$, and we define $b^r = \prod_{i=1}^k (b^{h_i})^{n_i} = \prod_{i=1}^k (b^{g_i})^{n_i}$, where $g_i \in G$. In this way B can be regarded as a multiplicatively written R-module. Since H is a finitely generated Abelian group, H is finitely presented (a group is finitely presented if it can be defined by finite number of generators subject to a finite number of relations), for instance see Corollary to Lemma 1.43 in [Rob72]. It follows that B is finitely generated as an R-module, by

Theorem 14.1.3 in [Rob82]. This is where commutative algebra comes into play in proving some critical results. Because of this connection we will use techniques and results from commutative algebra in chapter 5.

For notational convenience, throughout this chapter, we let R be a commutative ring and M an R-module.

3.2 The associated set of primes of a module

Recall that an ideal P is called a prime ideal of a ring R if R/P is an integral domain. Let M be an R-module, and $x \in M$. The *annihilator* $Ann(x)$ of x is the ideal consisting of all elements $r \in R$ such that $rx = 0$, that is,

$$Ann(x) = \{r \in R : rx = 0\}.$$

In general, the *annihilator* of M,

$$Ann(M) = \{r \in R : rM = 0\}.$$

A prime P of R is said to be associated to M if P is the annihilator of an element of M. The set of the primes associated to M is denoted by $Ass(M)$, that is,

$$Ass(M) = \{P : P = Ann(x), for\ some\ x \in M\},$$

where P is a prime ideal of R. From this we can see that P is an associated prime of M if and only if R/P is isomorphic to a submodule of M. Note that all the associated primes of M contain the annihilator of M.

Recall that a ring R is Noetherian if every ideal is finitely generated, or if it satisfies the ascending chain condition.

Note that if R is a Noetherian ring and M is a finitely generated non-zero R-module, then $Ass(M)$ is non-empty, by the following lemma.

Lemma 3.2.1 (Corollary 1, p. 262, [Bou72]) *Let R be a Noetherian ring and M a finitely generated R-module. Then the condition $M \neq 0$ is equivalent to $Ass(M) \neq \emptyset$.*

The following results will be used.

Theorem 3.2.2 (Theorem 1, p. 265, [Bou72]) *Let R be a Noetherian ring and M a finitely generated R-module. There exists a series $0 = M_0 < M_1 < \cdots < M_n = M$ of M such that $M_i/M_{i-1} \cong R/P_i$, where P_i is a prime ideal of R, $i = 1, 2, \cdots, n$.*

Theorem 3.2.3 (Corollary 3, p. 63, [Bou72]) *Let M be a finitely generated R-module. For an ideal J of R to be such that $JM = M$, it is necessary and sufficient that there exist $x \in J$ such that $(1 + x)M = 0$.*

Proposition 3.2.4 (Proposition 2, p. 262, [Bou72]) *Let M be a module over a ring R. Every maximal element of the set of ideals $Ann(x)$ of R, where x runs through the set of elements $\neq 0$ of M, belongs to $Ass(M)$.*

Theorem 3.2.5 (Krull Intersection Theorem, Theorem 74, [Kap70]) *Let R be a Noetherian ring, I an ideal in R, M a finitely generated R-module, and $N = \bigcap\limits_{n>0} MI^n$. Then $IN = N$.*

3.3 Primary decomposition

A submodule N of a module M is *primary* if $Ass(M/N)$ consists of just one prime ideal; if $Ass(M/N) = \{P\}$, we may say that N is P-primary; if $Ass(M) = \{P\}$ consists of just one prime ideal, we say that a module M is *coprimary* .

Let us recall the basic definition of a *primary decomposition* of a submodule.

Let R be a Noetherian ring, M an R-module, and let N be a submodule of M. A finite family $(Q_i)_{i\in I}$ of submodules of M which are primary with respect to M such that

$$N = \bigcap_{i\in I} Q_i$$

is called a *primary decomposition* of N in M.

Next we shall discuss some results giving certain uniqueness properties of a reduced primary decomposition.

Definition 3.3.1 *Let M be a module over a Noetherian ring and N a submodule of M. A* primary decomposition

$$N = \bigcap_{i\in I} Q_i$$

of N in M is called reduced if the following conditions are satisfied:

(a) *there exists no index* $i \in I$ *such that* $\bigcap\limits_{i \neq j} Q_j \subset Q_i$;

(b) *if* $Ass(M/Q_i) = \{P_i\}$, *the* $P_i (i \in I)$ *are distinct,* ([Bou72], p.270).

In other words, such a decomposition is said to be reduced if no Q_i can be omitted and if the associated primes of $M/Q_i, (1 \leq i \leq n)$, are all distinct. The next proposition will give us a necessary and sufficient condition for a primary decomposition to be reduced.

Proposition 3.3.2 (Proposition 4, p.271, [Bou72]) *Let* M *be a module over a Noetherian ring,* N *a submodule of* M, $N = \bigcap\limits_{i \in I} Q_i$ *a primary decomposition of* N *in* M *and, for all* $i \in I$, *let* $\{P_i\} = Ass(M/Q_i)$. *For this decompostion to be reduced, it is necessary and sufficient that the* P_i *be distinct and belong to* $Ass(M/N)$; *then*

$$Ass(M/N) = \bigcup_{i \in I} \{P_i\}$$
$$Ass(Q_i/N) = \bigcup_{j \neq i} \{P_j\}$$

for all $i \in I$.

The next theorem tells us if M is a finitely generated module over a Noetherian ring, then any submodule N of M has a primary decomposition.

Theorem 3.3.3 (Theorem 1, p. 270, [Bou72]) *Let* M *be a finitely generated module over a Noetherian ring and let* N *be a submodule of* M. *There exists a primary decomposition of* N *of the form* $N = \bigcap\limits_{P \in Ass(M/N)} Q_P$, *where for all* $P \in Ass(M/N)$, Q_P *is* P*-primary with respect to* M.

We may replace M by M/N and therefore suppose that $N = 0$. Then the primary decomposition of the zero submodule of M is $0 = M_1 \cap \cdots \cap M_n$, where $Ass(M) = \{P_1, \cdots, P_n\}$.

We conclude our discussion by giving the next theorem which relates the primes belonging to a *primary decomposition* with the associated primes discussed in the previous section.

Theorem 3.3.4 (Theorem 3.5, p. 423, [Lan02]) *Let* M *be a finitely generated module over a Noetherian ring* R. *The associated primes of* M *are precisely the primes which belong to the primary modules in a reduced primary decomposition of* 0 *in* M. *In particular, the set of associated primes of* M *is finite.*

3.4 The augmentation ideal

Recall that if G is a group, then the map $\varepsilon : G \longrightarrow \{1\}$ extends to a ring homomorphism

$$\varepsilon : \mathbb{Z}[G] \longrightarrow \mathbb{Z},$$

the *augmentation*, given by

$$\varepsilon(\sum_{g\in G} n_g g) = \sum_{g\in G} n_g,$$

where $n_g \in \mathbb{Z}$.

The kernel of ε is the *augmentation ideal*

$$I_G = \{\sum_{g\in G} n_g g : \sum_{g\in G} n_g = 0\},$$

which is an ideal of $\mathbb{Z}[G]$ generated as additive group by all the $g - 1 \neq 0$, that is, $I_G = \langle g-1, 1 \neq g \in G\rangle$.

If K is a normal subgroup of a group G, the relative augmentation ideal I_K of the integral group ring $\mathbb{Z}[G]$ is the kernel of a ring homomorphism of

$$\varphi : \mathbb{Z}[G] \longrightarrow \mathbb{Z}[G/K],$$

(that is, an epimorphism of Abelian groups which maps $g \mapsto gK$), which is an ideal I_K of $\mathbb{Z}[G]$ generated as additive group by all the $x - 1 \neq 0$, where $1 \neq x \in K$, that is, $I_K = \langle x-1, 1 \neq x \in K\rangle$ (see [Rob82], p. 322).

3.5 Some technical results

In this section we give some results, Lemma 3.5.1 and Corollary 3.5.2, which can be used as alternative method to prove Lemma F, (the second case, when the centre is trivial), of Chapter 5.

If G is a finitely generated group with a normal Abelian subgroup A such that G/A is nilpotent, then it is well known that the lower central series of G terminates at ω, that is, $\gamma_\omega(G) = \gamma_{\omega+1}(G) = \cdots$, where $\gamma_\omega(G) = \bigcap_{i=1}^{\infty} \gamma_i(G)$ denotes the ω^{th} term of the lower central series of G, (see 15.3.7, of [Rob82]).

Recall that a group G is residually nilpotent if and only if $\bigcap_{i=1}^{\infty} \gamma_i(G) = 1$. If G is a finitely generated metabelian group, then one can decide that when G is residually nilpotent by finding $\gamma_\omega(G) = \underset{i>0}{\cap} MI^i$, (Theorem 9.1, [BCR94]). This follows immediately by taking M to be $A = G'$ and I be the augmentation ideal of $\mathbb{Z}[G/A]$, where $M = A = G'$ is a finitely generated $\mathbb{Z}[G/A]$-module.

If $\gamma_\omega(G) = \underset{i>0}{\cap} MI^i = 0$, then in group theoretic terms $\gamma_\omega(G) = \underset{i>0}{\cap}[A, {}_iG] = 1$, and hence G is residually nilpotent.

In the next lemma we shall consider the case in which $\gamma_\omega(G) = \underset{i>0}{\cap} MI^i \neq 0$.

Lemma 3.5.1 *Let G be a finitely generated Abelian group. Let $R = \mathbb{Z}[G]$ be the integral group ring, and let K be a subgroup of G, and let M be a non-zero finitely generated R-module which is torsion free as Abelian group. Let I_K be the relative augmentation ideal of R. Assume that $\underset{i>0}{\cap} MI_K^i \neq 0$. If $Ass(M) = \{P\}$ and $R/P \cong M$ such that $I_K \neq P$, then $M = MI_K$.*

Proof. Let $N = \underset{i>0}{\cap} MI_K^i$. Clearly N is a submodule of M. Let $0 \neq a \in N$ be an element with maximal annihilator. Then we show that $Ann(a) = Q$ is a prime ideal.

If u and v are elements of R such that $uv \in Q$ and $u \notin Q$, then we shall show that v lies in Q. $u \notin Q$ implies that $au \neq 0$, but $auv = 0$, since $uv \in Q$. Observe that $Q = Ann(a) \subseteq Ann(au)$. But as Q is maximal, $Ann(au) \subseteq Ann(a) = Q$, which implies that $Ann(au) = Q$. Then v annihilates au, hence $v \in Q$, so that Q is prime.

Now one can see that $Q = Ann(a) \in Ass(N) \subseteq Ass(M) = \{P\}$. Since P is the only associated prime of M, and $Ass(N) \neq \emptyset$, it follows that $P = Q$, and so $Ann(a) = P$.

By Theorem 3.2.5, it follows that $a = ax$ for some $x \in I_K$. Then $a(1 - x) = 0$, and so $1 - x \in P$. Clearly we see that $1 - x$ annihilates M, that is, $(1 - x)M = 0$. Then by Theorem 3.2.3, it follows that $MI_K = M$. ■

The next Corollary is just a special case of Lemma 3.5.1.

Corollary 3.5.2 *Let G be a finitely generated Abelian group. Let $R = \mathbb{Z}[G]$ the integral group ring, and let M be a non-zero finitely generated R-module and torsion free as Abelian group. Let I be the augmentation ideal of R. Assume that $\underset{i>0}{\cap} MI^i \neq 0$. If $Ass(M) = \{P\}$ and $R/P \cong M$ such that $I \neq P$, then $M = MI$.*

Proof. The proof of this Corollary just follows from Lemma 3.5.1 by just defining I_K to be I. ■

Definition 3.5.3 *Let a ring R be an integral domain. An R-module M is torsion free if $0 \neq r \in R$ and $0 \neq x \in M$ implies $rx \neq 0$.*

We call an element x of an R-module M torsion free if $x \neq 0$ and $rx = 0$, then $r = 0$, where $r \in R$.

Lemma 3.5.4 *If G is a metabelian group such that G' is torsion free as $\mathbb{Z}[G/G']$-module, then $Z(G) = 1$.*

Proof. First we show that $Z(G) \leq G'$. Since G' is torsion free as $\mathbb{Z}[G/G']$-module, if $z \in Z(G)$, then $a^h = a$, for all $a \in G'$, where $h = zG'$. This implies that $1 = a^{-1}a^h = a^{h-1}$, for all $a \in G'$. Observe that $h-1$ acting on G' can be viewed as an element of the integral group ring $\mathbb{Z}[G/G']$. Since G' is torsion free as $Z[G/G']$-module, it follows that $h - 1 = 0$, i.e, $h = 1$ in $\mathbb{Z}[G/G']$ if and only if $z \in G'$. Hence $Z(G) \leq G'$.

Suppose that there exists $1 \neq z \in Z(G)$. Each $g \in G$ acts trivially on z by conjugation. But $Z(G) \leq G'$ and so $t-1$ acts trivially on z, for all $1 \neq t \in G/G'$. This is a contradiction, since G' is torsion free as $\mathbb{Z}[G/G']$-module. Hence $Z(G) = 1$. ■

We also need the next Lemma, a consequence of Theorem A of Wilson in [Wil70].

Lemma 3.5.5 *Let G be a finitely generated Abelian group, $\mathbb{Z}[G]$ the integral group ring and A a Noetherian $\mathbb{Z}[G]$-module. If K is a subgroup of G with finite index, then A is Noetherian as $\mathbb{Z}[K]$-module.*

CHAPTER 4

4. THE WIELANDT SUBGROUP OF INFINITE SOLUBLE GROUPS

Recall that the Wielandt subgroup $\omega(G)$ of a group G is the intersection of the normalisers of all the subnormal subgroups of G. This subgroup has been the subject of many papers since its introduction by Wielandt in [Wie58]. Wielandt obtained many basic properties of $\omega(G)$, showing in particular that all non-Abelian simple subnormal subgroups of G are contained in $\omega(G)$, and all minimal normal subgroups satisfying the minimal condition on normal subgroups are contained in $\omega(G)$ (see 13.3.2 and 13.3.7 of [Rob82]). The Wielandt subgroup has also been discussed in Camina [Cam70] and more recently in Casolo [Cas89], Bryce and Cossey [BC89], Ormerod [Orm90], Brandl, Franciosi and de Giovanni [BFdG90], and Cossey [Cos91]. Of course $\omega(G)$ contains the centre Z(G), and is non-trivial for a finite group, but for infinite groups the example of the infinite dihedral group shows that $\omega(G)$ can be trivial.

In this chapter we consider the Wielandt subgroup of an infinite soluble group with Max-n. The results will be used in the study of the group of all automorphisms fixing subnormal subgroups of certain finitely generated infinite soluble groups with Max-n, in particular, finitely generated metabelian groups and finitely generated Abelian-by-nilpotent groups in the following chapters.

Since the Wielandt subgroup plays a crucial role in the investigation of the group of all automorphisms fixing subnormal subgroups, one has to deal with the Wielandt subgroup in order to determine the group of all automorphisms fixing subnormal subgroups. In this chapter we also consider the action of the group $Aut_{sn}(G)$ on $F(G)$ (and hence on $\omega(G)$) when G is a finitely generated Abelian-by-nilpotent group.

In their paper [BFdG90](Theorem A), Brandl, Franciosi, and de Giovanni, showed that if G is a finitely generated soluble-by-finite group with finite Prüfer rank, then the Wielandt subgroup $\omega(G)$ is contained in the FC-centre of G. In this chapter we will give a sharper result on the Wielandt subgroup of an infinite soluble group with Max-n.

Note that a soluble group which satisfies Max-n is always finitely generated, (Theorem 5.33 in [Rob72], part 1).

The main results of this chapter are the following.

Theorem A *Let G be an infinite soluble group with Max-n. Then the Wielandt subgroup $\omega(G)$ of G is an Abelian group.*

This result yields the following Corollary.

Corollary B *Let G be a finitely generated Abelian-by-nilpotent group.*

(i) $\omega(G)$ *centralises* $F(G)$, *that is* $[F(G), \omega(G)] = 1$.

(ii) If B is an Abelian normal subgroup of G, then $B\omega(G)$ is an Abelian group.

The next result will tell us whether $C_G(\omega(G))$ has finite index in G, where G is a finitely generated Abelian-by-nilpotent group.

For a polycyclic group G, Cossey [Cos91], (Theorem 1(i)), showed that $C_G(\omega(G))$ has finite index in G.

On the other hand, as an immediate consequence of Theorem A of [BFdG90], and as slight generalisation of Cossey's result Theorem 1(i) in [Cos91], Brandl, Franciosi, and de Giovanni [BFdG90] proved that, if G is a polycyclic-by-finite group, then $G/C_G(\omega(G))$ is finite.

We will also give a similar result for a finitely generated Abelian-by-nilpotent group.

Theorem C *Let G be a finitely generated Abelian-by-nilpotent group. Then $C_G(\omega(G))$ has finite index in G.*

Remark. From Theorem C we see easily $\omega(G)$ is finitely generated. Since G satisfies Max-n and $C_G(\omega(G))$ has finite index in G, $C_G(\omega(G))$ satisfies Max-n, by Theorem 5.31 of Wilson in [Rob72]. Since every subgroup of $Z(C_G(\omega(G)))$ is normal in $C_G(\omega(G))$, it follows that $Z(C_G(\omega(G)))$ is finitely generated, but

$$\omega(G) \leq Z(C_G(\omega(G))),$$

and hence $\omega(G)$ is finitely generated.

The next Theorem will tell us the action of the group $Aut_{sn}(G)$ on $F(G)$ (and hence on $\omega(G)$), when G is a finitely generated Abelian-by-nilpotent group. We remind the reader that we only consider non-Abelian infinite groups.

Theorem D *Let G be a finitely generated Abelian-by-nilpotent group and set $\Gamma = Aut_{sn}(G)$.*

(i) If G' is finite, then Γ has a subgroup of index at most two centralising $F(G)$.

(ii) If G' is infinite, then Γ centralises $F(G)$, and hence $[\Gamma, \omega(G)] = 1$.

4.1 Proofs

Recall that the Wielandt subgroup $\omega(G)$ of a group G is always a T-group and always contains the centre $Z(G)$ of G. If G is a soluble group, then $\omega(G)$ is a metabelian T-group, as every soluble T-group is metabelian by Theorem 2.3.1 of Robinson in [Rob64]. The proof of Theorem A depends on the classification of soluble T-groups of Robinson [Rob64].

Proof of Theorem A. Let G be an infinite soluble group with Max-n and let $W = \omega(G)$. Then W is a metabelian T-group.

Since the presence of elements of infinite order in a soluble T-group may affect the structure of the group, it is meaningful to treat separately periodic and non-periodic soluble T-groups. For this reason we shall split the proof into two cases, W is a periodic soluble T-group, and W is a non-periodic soluble T-group.

(i) W is a periodic soluble T-group.

If $W' \neq 1$, then W' is a divisible Abelian group by Theorem 4.3.1 of Robinson [Rob64]. This is a contradiction, since G satisfies *Max-n*. Hence $W' = 1$. The result follows.

(ii) W is a non-periodic soluble T-group.

Now suppose that $W' \neq 1$. Then $C_W(W') = F(W)$, the Fitting subgroup of W, by Lemma 2.2.2 of Robinson [Rob64]. Since every subgroup of a nilpotent group is subnormal, a nilpotent T-group has each of its subgroups normal and so is a Dedekind group, (a group in which every subgroup is normal), and hence $F(W)$ is a Dedekind group.

Throughout the proof, we set $F = F(W)$. In this case also we shall consider when F is a periodic, and F is a non-periodic normal subgroup of W.

(a) Assume that F is non-periodic. Then F is Abelian, as a Dedekind group with elements of infinite order is Abelian, and hence W is a soluble T-group of type 1, (a soluble T-group H is of type 1 if H is non-Abelian and $C = C_H(H') = F(H)$ is non-periodic, (section 2.5, [Rob64])). Let $y \in W \backslash F$. Since W is a soluble T-group and F is a non-periodic Abelian normal subgroup of W, then y induces a non-trivial power automorphism on F, that is, $x^y = x^{-1}$, for all x in F, (13.4.3 (i), [Rob82]). Now $x^{y^2} = x$, which gives that $y^2 \in F$, since $C_W(F) = F$. Now put $a = y^2$, since $a \in F$ of course $a^y = a^{-1}$. It follows that $aa^y = y^2(y^2)^y = y^4 = 1$, and so y has order 4.

On the other hand, every soluble T-group of type 1 has the structure

$$\langle y^2, F^2 \rangle = \langle y^2, F^4 \rangle,$$

where $y \in W \backslash F$, and $x^y = x^{-1}$, for all x in F, (Theorem 3.1.1, [Rob64]). Let T be the periodic subgroup of F. As F is Abelian, clearly T is characteristic in F, and so T is normal in G. Observe that since $y^4 = 1$, then $y^2 \in T$. Then $\langle y^2, F^2 \rangle / T = \langle y^2, F^4 \rangle / T$, which gives that $F^2/T = F^4/T$. This is a contradiction, since $F^2/T = F^4/T \lhd F/T$ are torsion free Abelian groups, and so $F = T$, which is impossible as F is non-periodic by the assumption above. Therefore W is an Abelian group.

(b) Assume that F is periodic. Then W is a soluble T-group of type 2, (a soluble T-group H is of type 2 if H is non-Abelian and non-periodic, and $C_H(H')$ is periodic). At once it follows from Robinson's Theorem 4.3.1 in [Rob64] that W' is a divisible Abelian group. This is a contradiction, since G satisfies Max-n. Therefore $W' = 1$, and hence W is an Abelian group.

This proves in any case that W is an Abelian group. ■

To prove Corollary B we need the following lemma.

Lemma 4.1.1 *Let G be both Abelian-by-nilpotent and polycyclic. Then $\omega(G)$ centralises $F(G)$, that is, $[\omega(G), F(G)] = 1$.*

Proof. Let G be both Abelian-by-nilpotent and polycyclic. Then we first show that $F(G)$ is a non-periodic nilpotent group.

Suppose that $F(G)$ is periodic. Since G is polycyclic, $F(G)$ is finitely generated and so finite, as a finitely generated soluble periodic group is finite, (see 5.4.11 of [Rob82]). Moreover, its centraliser $C_G(F(G))$ has finite index in G and also is finite, as $C_G(F(G)) = Z(F(G))$, (see 5.4.4(ii) of Robinson [Rob82])), hence it follows easily that G is finite. This is a contradiction, since G is an infinite group.

This proves that $F(G)$ is always a non-periodic nilpotent group in an infinite polycyclic group.

We know that $\omega(G)$ is an Abelian group, by Theorem A, and contained in $F(G)$. Since $F(G)$ is a non-periodic nilpotent group, $\omega(G)$ acts as a group of power automorphisms on $F(G)$. If $F(G)$ is not Abelian, it has no non-trivial power automorphisms, by Lemma 2.5.3(ii), and so $\omega(G)$ centralises $F(G)$. If $F(G)$ is Abelian, then $\omega(G)$ centralises $F(G)$, as $\omega(G)$ is contained in $F(G)$. Hence in any case it follows that $[\omega(G), F(G)] = 1$. ■

We are now in a position to prove Corollary B.

Proof of Corollary B.
(i) Let G be a finitely generated Abelian-by-nilpotent group. If $F(G)$ is Abelian, then clearly $[F(G), \omega(G)] = 1$, as $\omega(G)$ is Abelian by Theorem A and contained in $F(G)$.

We shall consider when $F(G)$ is a non-periodic nilpotent group, and $F(G)$ is a periodic nilpotent group.

(a) Suppose that $F(G)$ is a non-periodic nilpotent group.

Since every subgroup of $F(G)$ is subnormal in G, $\omega(G)$ acts as a group of power automorphisms on the non-periodic nilpotent group $F(G)$.

If $F(G)$ is not Abelian, then $F(G)$ has no non-trivial power automorphisms, by Lemma 2.5.3(ii), and so $\omega(G)$ centralizes $F(G)$, and hence $[F(G), \omega(G)] = 1$.

(b) Suppose that $F(G)$ is a periodic nilpotent group, that is, $F(G) \leq \tau(G)$.

We show that $[\omega(G), F(G)] = 1$. Suppose, on the contrary, that

$$[\omega(G), F(G)] \neq 1.$$

Then there exist $u \in \omega(G)$ and $x \in F(G)$ such that $[u, x] \neq 1$. Since $G \in \mathfrak{G} \cap \mathfrak{A}\mathfrak{N}$, G is residually finite, by Theorem 2.2.2(i)(b), so there exists $K \lhd G$ such that

$[u, x] \notin K$ and K has finite index in G. Now K, as subgroup of finite index in a finitely generated group, is finitely generated. Moreover, K is non-periodic, and so infinite, as G is a non-periodic group.

Observe that G/K is finite and soluble, and hence it is polycyclic.

On the other hand, since $K \in \mathfrak{G} \cap \mathfrak{AN}$, clearly $K/F(K)$ is polycyclic. Now since $K/F(K)$ and G/K are polycyclic groups, it follows easily that $G/F(K)$ is also a polycyclic group.

Since $F(K)$ is periodic, clearly $G/F(K)$ is an infinite polycyclic group, and so $F(G/F(K))$ is a non-periodic nilpotent group. But $\omega(G/F(K))$ centralises $F(G/F(K))$, by Lemma 4.1.1, that is

$$[\omega(G/F(K)), F(G/F(K))] = 1.$$

Since $\omega(G)F(K)/F(K) \leq \omega(G/F(K))$ and $F(G)/F(K) \leq F(G/F(K))$, it follows easily that

$$[\omega(G)F(K)/F(K), F(G)/F(K)] = 1,$$

and so $[\omega(G), F(G)] \leq F(K) \leq K$. This is a contradiction, since $[u, x] \notin K$. Hence $[\omega(G), F(G)] = 1$.

(ii) Since B is an Abelian normal subgroup of G, B is contained in $F(G)$, and so $[\omega(G), B] \leq [\omega(G), F(G)] = 1$, and hence $B\omega(G)$ is an Abelian group. ■

Now we can see in the following Corollary that if A is a maximal Abelian normal subgroup of a finitely generated metabelian group G, then the Wielandt subgroup $\omega(G)$ of G is contained in A, which is a direct consequence of Corollary B and Corollary 2.4.4.

Corollary 4.1.2 *Let G be a finitely generated metabelian group, and let A be a maximal Abelian normal subgroup of G. Then $\omega(G) \leq A$.*

Proof. Since $\omega(G)$ centralises $F(G)$, by Corollary B, $\omega(G)$ also centralises A, that is $[\omega(G), A] = 1$. It follows that $\langle \omega(G), A\rangle$ is Abelian and $A\omega(G) = A$. Hence the result follows. ■

Proof of Theorem C. Let G be a finitely generated Abelian-by-nilpotent group. Since $G \in \mathfrak{G} \cap \mathfrak{AN}$, there exists a normal subgroup H of finite index in G such that H is residually finite-nilpotent by Theorem 2.6.3. Put $K = \omega(G)H$.

Then by Lemma 2.6.1, K is residually finite-nilpotent, and so $\omega(K)$ is contained in $Z_2(K)$, that is,

$$\omega(K) \leq Z_2(K),$$

by Lemma 2.6.2. We split the proof into two cases, $R(K) = 1$, and $R(K) \neq 1$.

(i) $R(K) = 1$.

As K is residually finite-nilpotent, if $\tau(K) = 1$, then $\omega(K) = Z(K)$, by Lemma 2.6.2(ii). But

$$\omega(G) \leq \omega(K) = Z(K),$$

and so $[\omega(G), K] = 1$. Hence it follows that $K \leq C_G(\omega(G))$, and so $C_G(\omega(G))$ has finite index in G.

Suppose that $\tau(K) \neq 1$. Since $R(K) = 1$, $\tau(K)$ has no non-trivial finite normal subgroup, and so

$$\tau(K) \cap Z(K) = 1. \tag{4.1}$$

Observe that since $K/\tau(K) \in \mathfrak{G} \cap \mathfrak{A}\mathfrak{N}$, there exists a normal subgroup $L/\tau(K)$ of finite index in $K/\tau(K)$ such that $L/\tau(K)$ is residually finite-nilpotent by Theorem 2.6.3, and so L has finite index in K. We can assume that $\omega(K) \leq L$, by Lemma 2.6.1. Since $\tau(L/\tau(K)) = 1$ and $L/\tau(K)$ is residually finite-nilpotent, from Lemma 2.6.2(ii), it follows that

$$\omega(L/\tau(K)) = Z(L/\tau(K)),$$

which implies that

$$[\omega(L), L] \leq \tau(K). \tag{4.2}$$

On the other hand, since K is residually finite-nilpotent, $\omega(K) \leq Z_2(K)$, which gives us

$$[\omega(L), L] \leq [\omega(K), K] \leq Z(K). \tag{4.3}$$

Now from (4.1), (4.2), and (4.3), it follows that

$$[\omega(G), L] \leq [\omega(K), L] \leq [\omega(L), L] \leq Z(K) \cap \tau(K) = 1, \tag{4.4}$$

and so $L \leq C_G(\omega(G))$.

Hence $C_G(\omega(G))$ has finite index in G in this case.

(ii) $R(K) \neq 1$.

Clearly $K/R(K)$ has no finite normal subgroup, that is,

$$R(K/R(K)) = 1,$$

and so $\tau(K/R(K)) \cap Z(K/R(K)) = 1$.

Since $K/R(K) \in \mathfrak{G} \cap \mathfrak{A}\mathfrak{N}$, there exists a normal subgroup $M/R(K)$ of finite index in $K/R(K)$ such that $M/R(K)$ is residually finite-nilpotent by Theorem 2.6.3. We can assume that $\omega(K) \leq M$, by Lemma 2.6.1.

Now by (i) and (4.4) above, it follows that

$$[M/R(K), \omega(K/R(K))] = 1,$$

and so $[M/R(K), \omega(K)R(K)/R(K)] = 1$, which implies that $[M, \omega(K)] \leq R(K)$. Since $\omega(G) \leq \omega(K)$, it follows that

$$[M, \omega(G)] \leq [M, \omega(K)] \leq R(K).$$

As $M \in \mathfrak{G} \cap \mathfrak{A}\mathfrak{N}$, M is residually finite, by Theorem 2.2.2$(i)(b)$, for $x \in R(K)$ there exists $N_x \lhd M$ such that $1 \neq x \notin N_x$ and M/N_x is finite. Now put $N = \bigcap_{1 \neq x \in R(K)} N_x$, and note that N has finite index in K and K has finite index in G, and so N has finite index in G. Observe that $N \cap R(K) = 1$. But we know that

$$[\omega(G), N] \leq [\omega(K), M] \leq R(K). \tag{4.5}$$

On the other hand,

$$[\omega(G), N] \leq N. \tag{4.6}$$

Now from (4.5), and (4.6), it follows that

$$[\omega(G), N] \leq N \cap R(K) = 1,$$

and so $N \leq C_G(\omega(G))$. Thus $C_G(\omega(G))$ has finite index in G in this case as well.

This proves in any case that $C_G(\omega(G))$ has finite index in G. ∎

Next we show that the action of the group of automorphisms fixing subnormal subgroups of a finitely generated Abelian-by-nilpotent group on the Fitting subgroup and also the Wielandt subgroup.

Proof of Theorem D. Let G be a finitely generated Abelian-by-nilpotent group and set $\Gamma = Aut_{sn}(G)$.

(i) Observe that since G' is finite and G is finitely generated, $Z(G)$ has finite index in G, by Corollary 5.41 of [Neu51], and hence G is centre-by-finite.

Since G is an infinite group and G' is finite, G/G' is infinite.

We first show that Γ has a subgroup of index at most two centralising G/G'.

Let

$$\Psi = C_{\Gamma}(G/G') = \{\alpha \in \Gamma : g^{\alpha}G' = gG', \ for\ all\ gG' \in G/G'\}.$$

Now if $1 \neq \delta \in \Gamma \backslash \Psi$, then δ acts as inversion on G/G' and hence has order two. Since the factor group Γ/Ψ is isomorphic to a group of power automorphisms of the non-periodic Abelian group G/G', it follows that Γ/Ψ has order two, and so $\Gamma = \Psi \rtimes \langle \delta \rangle$. Hence Ψ has at most index two in Γ.

Next we show that $[\Psi, F(G)] = 1$.

Note that since G is a centre-by-finite infinite group, $Z(G)$ is a non-periodic Abelian group, and so $F(G)$ is a non-periodic nilpotent group.

We divide the proof into two cases, when $F(G)$ is non-Abelian nilpotent, and when $F(G)$ is Abelian.

(a) Suppose that $F(G)$ is non-Abelian nilpotent.

Let $\beta \in \Psi$. Since $F(G)$ is non-Abelian nilpotent, β acts trivially on $F(G)$, by Lemma 2.5.3(ii), and so β centralises $F(G)$. Hence $[F(G), \beta] = 1$.

(b) Suppose that $F(G)$ is non-periodic Abelian and β acts as inversion on $F(G)$.

Since $F(G)$ is non-periodic Abelian, we may choose $a \in F(G)$ of infinite order, and then β inverts a, that is

$$(a)^{\beta} = a^{-1}. \tag{4.7}$$

Since β acts trivially on G/G', it also acts trivially on aG', that is $a^{\beta} \in aG'$, and so

$$a^{\beta} = ay, \tag{4.8}$$

for some $y \in G'$.

From (4.7) and (4.8), it follows that $a^{-1} = ay$, and so $a^{-2} = y$. This is a contradiction, since y is an element of finite order and a is an element of infinite order. Hence β acts trivially on $\langle a \rangle$ for any $a \in F(G)$ of infinite order.

Since $F(G)$ can be generated by elements of infinite order, β acts trivially on $F(G)$, and so $[F(G), \Psi] = 1$.

(ii) Now G' is infinite and we show that $[\Gamma, F(G)] = 1$.

We divide the proof into two by considering when $F(G)$ is a non-periodic nilpotent group, and when $F(G)$ is a periodic nilpotent group. Set $\Gamma = Aut_{sn}(G)$ and let $\alpha \in \Gamma$.

(1) Suppose that $F(G)$ is a non-periodic nilpotent group.

Since every subgroup of $F(G)$ is subnormal in G, α acts as a group of power automorphisms on $F(G)$. If $F(G)$ is not Abelian, it has no non-trivial power automorphisms, by Lemma 2.5.3(ii), and so α centralises $F(G)$. Hence $[F(G), \Gamma] = 1$.

If $F(G)$ is Abelian and non-periodic, then α acts on $F(G)$ either as inversion or the identity. If α acts as the identity on $F(G)$, then there is nothing to prove.

We consider when $\tau(F(G)) = 1$, and when $\tau(F(G)) \neq 1$.

(a) Suppose that $\tau(F(G)) = 1$ and α acts as inversion on $F(G)$.

Since $G \in \mathfrak{G} \cap \mathfrak{A}\mathfrak{N}$, there exists a normal subgroup K of finite index in G such that K is residually finite-nilpotent, by Theorem 2.6.3. We can assume that $F(G) \leq K$, by Lemma 2.6.1.

Now we consider when K is Abelian, and when K is non-Abelian.

Suppose that K is Abelian.

Then $K \leq F(G)$, and so $F(G) = K$, and hence G is polycyclic. It follows that G is an Abelian-by-finite polycyclic group, and so $\omega(G)/Z(G)$ is finite, by Theorem B of [BFdG90]. As $\omega(G)/Z(G)$ is finite, there exists $n \in \mathbb{Z}$ such that $u^n Z(G) = Z(G)$, for all $u \in \omega(G)$. It follows that $u^n \in Z(G)$. Since $\omega(G)$ is Abelian,

$$[u^n, g] = [u, g]^n = 1,$$

for all $g \in G$, which implies that $[u, g] = 1$, as $\tau(\omega(G)) = 1$, and so $u \in Z(G)$, which yields that $\omega(G) = Z(G)$.

Now let $h, g \in G$. Then

$$[h, g]^\alpha = [hu, gv] = [h, g]$$

for some $u, v \in \omega(G)$. Hence α acts trivially on G'. Since G' is infinite and polycyclic, $F(G')$ is infinite.

Now we show that $F(G') \neq 1$. Suppose that $F(G') = 1$. Then $G' \cap F(G) = 1$, which implies that $F(G) \leq Z(G)$. Thus $Z(G)$ has finite index in G and it follows that G' is finite, by Theorem 5.3 of [Neu51]. This is a contradiction, since G' is infinite. Hence $F(G') \neq 1$. Note that $F(G') \neq F(G)$ or $F(G')$ would be fixed by α.

Let $1 \neq y \in F(G')$ and let $1 \neq x \in F(G) \setminus F(G')$. Now since $xy \in F(G)$ and α acts as inversion on $F(G)$,

$$(xy)^\alpha = (xy)^{-1} = y^{-1}x^{-1} = x^{-1}y^{-1},$$

as $F(G)$ is Abelian.

On the other hand, since α acts trivially on $F(G')$,

$$(xy)^\alpha = x^\alpha y^\alpha = x^{-1}y.$$

It follows that $x^{-1}y = x^{-1}y^{-1}$, which implies that $y^2 = 1$, and so $y = 1$, as y is element of infinite order. This is a contradiction, since $y \neq 1$. Hence α acts trivially on $F(G)$, and so $[F(G), \Gamma] = 1$.

Suppose that K is non-Abelian.

Then

$$\omega(G) \leq \omega(K) \leq Z_2(K),$$

by Lemma 2.6.2(i), but $\omega(K) \leq F(G)$, and so $\omega(K)$ is torsion free, which implies that $\omega(K) = Z(K)$, by Lemma 2.6.2(ii). Now let $x, y \in K$. Then

$$[x, y]^\alpha = [xu, yv] = [x, y],$$

for some $u, v \in \omega(G)$. Hence α acts trivially on K', but $F(K') \leq F(G)$ and $\tau(F(K')) = 1$. Let $1 \neq b \in F(K')$. Let $1 \neq a \in F(G) \setminus F(K')$. Now since $ab \in F(G)$ and α acts as inversion on $F(G)$,

$$(ab)^\alpha = (ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1},$$

as $F(G)$ is Abelian.

On the other hand, since α acts trivially on $F(K')$,

$$(ab)^\alpha = a^\alpha b^\alpha = a^{-1}b.$$

It follows that $a^{-1}b = a^{-1}b^{-1}$, which implies that $b^2 = 1$, and so $b = 1$, as b is element of infinite order. This is a contradiction, since $b \neq 1$. Hence α acts trivially on $F(G)$, and so $[F(G), \Gamma] = 1$.

(b) Suppose that $\tau(F(G)) \neq 1$.

Since $F(G)$ is non-periodic Abelian, α acts as inversion on $F(G)$. Let T be the torsion subgroup of $F(G)$, and so $F(G) = T \times N$, where N is torsion free. Let $m = exp(T)$, the exponent of T. Then $F(G)^m$ is a torsion free Abelian normal subgroup of G.

Since $\tau(F(G)^m) = 1$, α acts trivially on $F(G)^m$, by (a) above.

As T is torsion Abelian, α acts as a power automorphism on T, that is for $t \in T$, $t^\alpha = t^n$, for some $n \in \mathbb{Z}$.

Now let $1 \neq x \in F(G)^m$ and $1 \neq t \in T$, and so $(xt) \in F(G)$ and has infinite order. Then clearly $(xt)^\alpha \neq (xt)$ and so $(xt)^\alpha = (xt)^{-1}$, by our supposition. But since $F(G)$ is Abelian,

$$(xt)^\alpha = (xt)^{-1} = x^{-1}t^{-1}. \tag{4.9}$$

On the other hand,

$$(xt)^\alpha = x^\alpha t^\alpha = xt^n \tag{4.10}$$

From (4.9) and (4.10), it follows that $x^{-1}t^{-1} = xt^n$, and so $x^{-2} = t^{n+1}$. This is a contradiction, since x is an element of infinite order. Hence α acts trivially on $F(G)$, and so $[F(G), \Gamma] = 1$.

(2) Suppose that $F(G)$ is a periodic nilpotent group, that is, $F(G) \leq \tau(G)$. We shall show that $[F(G), \Gamma] = 1$. Suppose, on the contrary, that

$$[F(G), \Gamma] \neq 1.$$

Since $G \in \mathfrak{S} \cap \mathfrak{A}\mathfrak{N}$, there exists a normal subgroup H of finite index in G such that H is residually finite-nilpotent by Theorem 2.6.3. We can assume that $F(G) \leq H$, by Lemma 2.6.1. Since H has finite index in G and contains $F(G)$, H is non-Abelian.

Suppose there exist $y \in F(G)$ and $\beta \in \Gamma$ such that $[y, \beta] \neq 1$. Since H is residually finite, by Theorem 2.2.2(i)(b), there exists $N \lhd G$ such that $[y, \beta] \notin N$ and N has finite index in H and we can also choose H/N is non-Abelian but nilpotent.

Now N is finitely generated, moreover, N is non-periodic, and so infinite, as G is infinite. Observe that H/N is soluble and finite, and so it is polycyclic.

On the other hand, since $H \in \mathfrak{G} \cap \mathfrak{A}\mathfrak{N}$, clearly $H/F(G)$ is a polycyclic group. We know that

$$F(G) \cap N = F(N),$$

by Lemma 2.4.2, and hence $N/F(N) \cong NF(G)/F(G)$ is polycyclic. Now as $N/F(N)$ and H/N are polycyclic groups, it follows that $H/F(N)$ is also a polycyclic group. Since $F(N)$ is periodic nilpotent, clearly $H/F(N)$ is a non-periodic polycyclic group, and so $F(H/F(N))$ is a non-periodic nilpotent group, and hence β acts trivially on $F(H/N)$, by Lemma 2.5.3(ii). But since $F(G) \leq H$, $F(G)/F(N) \leq F(H/F(N))$, which implies that

$$[F(G), \Gamma] \leq F(N) \leq N.$$

This is a contradiction, since $[y, \beta] \notin N$. Hence $[F(G), \Gamma] = 1$, and the result follows. ■

CHAPTER 5

5. PROOF OF MAIN RESULT

This chapter presents the proof of the main result, Theorem E. Recall that we only consider **non-Abelian infinite groups**. Its proof divided into two parts. We first consider the case that G has trivial torsion radical ($\tau(G) = 1$) and then the case that G has non-trivial torsion radical ($\tau(G) \neq 1$). The methods used are quite different for each case. In the trivial torsion radical case the methods of the proof depend heavily upon the techniques and results from commutative algebra which we have discussed in chapter 3, while in the non-trivial torsion radical case it depends on group theoretic techniques alone. In section 1 we prove that if $\tau(G) = 1$ then the Wielandt subgroup is just the centre of G and that $Aut_{sn}(G) = 1$ (Lemma F), while in section 2 we will consider the non-trivial torsion radical case when G' is infinite and prove that the group $Aut_{sn}(G)$ is a finite Abelian group, and when G' is finite and prove that $Aut_{sn}(G)$ is a finite metabelian group.

In general, we prove that the structure of the group $Aut_{sn}(G)$ of a finitely generated metabelian group G is a finite metabelian group, controlled by R(G), the finite radical of G, and $Z(G)$ has finite index in $\omega(G)$.

We restate the main result of this work, Theorem E.

Theorem E *Let G be a finitely generated metabelian group.*

(a) If G' is infinite, then

  (i) The group $Aut_{sn}(G)$ is a finite Abelian group;

  (ii) $[G, Aut_{sn}(G)] \leq R(G)$; and

  (iii) $Z(G)$ has finite index in $\omega(G)$.

(b) If G' is finite, then the group $Aut_{sn}(G)$ is a finite metabelian group.

In the investigation of the group $Aut_{sn}(G)$ of a finitely generated metabelian group G an important role is played by the Wielandt subgroup $\omega(G)$ of G. In

Chapter 4 we showed that $\omega(G)$ is an Abelian group. More precisely, in this chapter we show that $\omega(G)$ is central ($\omega(G) = Z(G)$), when the finite radical of G is trivial ($R(G) = 1$).

5.1 Proof of Theorem E, with trivial torsion radical

Note that if $\tau(G) = 1$, G' is not finite since G is a non-Abelian group.

Before we begin to prove the critical lemma, Lemma F, we need the following lemma.

Lemma 5.1.1 *Let G be a metabelian and polycyclic group such that $\tau(G) = 1$. Then $\omega(G) = Z(G)$ and $Aut_{sn}(G) = 1$.*

Proof. Let G be both metabelian and polycyclic. Note that since $\tau(G) = 1$, G' is infinite. Since G is nilpotent-by-Abelian and polycyclic, it follows that $Z(G)$ has finite index in $\omega(G)$ by Theorem 1 of Cossey [Cos91]. As $\omega(G)/Z(G)$ is finite, there exists $n \in \mathbb{Z}$ such that $x^n Z(G) = Z(G)$, for all $x \in \omega(G)$. It follows that $x^n \in Z(G)$. Since $\omega(G)$ is Abelian,

$$[x^n, g] = [x, g]^n = 1,$$

for all $g \in G$, which implies that $[x, g] = 1$, as $\tau(\omega(G)) = 1$, and so $x \in Z(G)$, which yields that $\omega(G) = Z(G)$. We know that $Aut_{sn}(G)$ acts trivially on $G/\omega(G)$ and $\omega(G)$, by Lemma 2.8.7 and Lemma D respectively, and so stabilises the series $1 \leq \omega(G) \leq G$. Hence $Aut_{sn}(G)$ is Abelian by Theorem 2.7.2. Now $g^\alpha = gu$, for all $g \in G$, and for some $u \in \omega(G)$, $\alpha \in Aut_{sn}(G)$. Then if we repeatedly apply α to g we get $g^{\alpha^n} = gu^n$ for every positive integer n. It follows that every non-identity element of $Aut_{sn}(G)$ is of infinite order, since $\tau(\omega(G)) = 1$. Therefore $Aut_{sn}(G)$ is a torsion free Abelian group.

On the other hand,

$$Aut_{sn}(G) \cap Inn(G)$$

has finite index in $Aut_{sn}(G)$, by Theorem 2.8.3, and

$$Aut_{sn}(G) \cap Inn(G) = 1,$$

as $\omega(G) = Z(G)$, it follows that $Aut_{sn}(G)$ is finite, which is a contradiction. Therefore $Aut_{sn}(G) = 1$. ∎

Note that by Lemma 5.1.1 we can now assume that G/G' is infinite since G/G' finite gives G polycyclic.

The next result gives a proof of Theorem E for the trivial torsion radical case. We prove this result by means of module theoretic results.

Lemma F *Let G be a finitely generated metabelian group with $\tau(G) = 1$. Then there are no non-trivial automorphisms that fix every subnormal subgroup of G setwise, that is, $Aut_{sn}(G) = 1$, and $\omega(G) = Z(G)$.*

Proof. Let G be a finitely generated metabelian group with G/G' infinite and A a maximal Abelian normal subgroup of G containing G'. Then $\omega(G) \leq A$, by Corollary 4.1.2 and A is torsion free.

Let $R = \mathbb{Z}[G/A]$ be the integral group ring of G/A.

Suppose that $1 \neq \alpha \in Aut_{sn}G$ and there exists $g \in G$ such that $g^\alpha = gu$ for some $1 \neq u \in \omega(G)$. Obviously $\omega(G)$ is a torsion free abelian group, and so $\langle u \rangle$ is infinite cyclic. Put

$$\Phi = \{N \lhd G : N \cap \langle u \rangle = 1\}.$$

Then Φ is non-empty, as $1 \in \Phi$. Since G satisfies Max-n, Φ contains a maximal element, M say. Consider $\langle u \rangle M/M \leq G/M$ and note that $\langle u \rangle M/M$ is infinite cyclic.

(a) If $L \lhd G$ and $L \neq M$ and $L \geq M$, then $L/M \cap \langle u \rangle M/M \neq 1$.

Suppose, on the contrary, that $L/M \cap \langle u \rangle M/M = 1$. Then $L \cap \langle u \rangle M \leq M$, and so $L \cap \langle u \rangle \leq M \cap \langle u \rangle = 1$, which implies that $L \cap \langle u \rangle = 1$. This is a contradiction, since $L \notin \Phi$. Therefore $L/M \cap \langle u \rangle M/M \neq 1$.

(b) $\tau(G/M) = 1$.

Suppose, on the contrary, that $\tau(G/M) = K/M \neq 1$. Then K/M is periodic normal subgroup of G/M. Since $M \leq K$,

$$K/M \cap \langle u \rangle M/M \neq 1.$$

It follows that $u^n M \in K/M$, for some $n \in \mathbb{Z}$. This is a contradiction, since $u^n M$ is of infinite order. Hence $\tau(G/M) = 1$.

By note after Lemma 5.1.1 $(G/M)/(G/M)'$ must be infinite and since

$$(gM)^\alpha \neq gM$$

and $(uM)^\alpha = uM$, G/M is not Abelian and $(G/M)'$ is not finite.

As G/M is a finitely generated metabelian group and $\tau(G/M) = 1$, clearly G/M satisfies the hypothesis of Lemma F, and α acts as a non-trivial automorphism of G/M. Hence

$$Aut_{sn}(G/M) \neq 1.$$

Hence we may assume that $M = 1$ and so by (a) above, we have $L \cap \langle u \rangle \neq 1$ for $L \lhd G$. Now we have that if $L \leq A$,

$$L \cap \langle u \rangle \leq A \cap \langle u \rangle \neq 1.$$

We are now in a position to use module theoretic results from commutative algebra.

Since R is a commutative Noetherian ring and A is a finitely generated R-module, there exists a primary decomposition of the zero submodule of A, that is,

$$A_1 \cap A_2 \cap ... \cap A_n = 0,$$

for submodules $A_1, A_2, \cdots, A_n$ of A, by Theorem 3.3.3. Then it is obvious that the R-submodules of A coincide with normal subgroups of G which are contained in A. Thus in group-theoretic terms, there exist normal subgroups N_i of G such that

$$N_1 \cap N_2 \cap \cdots \cap N_n = 1,$$

where $N_i \leq A$ and $N_i \lhd G$, for each $i = 1, 2, 3, \cdots, n$.

Since $N_i \cap \langle u \rangle \neq 1$, $\langle u^{k_i} \rangle \leq N_i$ for some k_i, and for all $i = 1, 2, 3, \cdots, n$, and then $\langle u^{k_i} \rangle \leq N_i$, for each i. It follows that $\langle u^h \rangle \leq \cap N_i = 1$, where $h = \prod k_i$, and hence $u^h = 1$. But since u is of infinite order, $u = 1$. This is a contradiction, since $u \neq 1$. Therefore A is coprimary.

Now R is a commutative Noetherian ring and A is a finitely generated R-module and so there exists a series

$$0 = A_0 \leq A_1 \leq \cdots \leq A_n = A$$

such that

$$A_i/A_{i-1} \cong R/P_i,$$

by Theorem 3.2.2, where P_i, is a prime ideal of R, for each $i = 1, 2, \cdots, n$. Since A is coprimary, it follows that

$$A_i/A_{i-1} \cong R/P,$$

where P is the only prime ideal associated to A, that is, $Ass(A) = \{P\}$.

We show that $u \in A_1$. Since $A_1 \cap \langle u \rangle \neq 0$, additively, $ku \in A_1$, for some $0 \neq k \in \mathbb{Z}$. But $A_2/A_1 \cong A_1$ and is torsion free as Abelian group. It follows that $k(u + A_1) = 0$, and so $u + A_1 = 0$, which implies that $u \in A_1$.

Now we split the proof into two cases, $Z(G) \neq 1$, and $Z(G) = 1$.

(i) $Z(G) \neq 1$.

Then $Z(G) \cap \langle u \rangle \neq 1$. Now $u^k \in Z(G)$, for some $k \in \mathbb{Z}$. But

$$[u^k, g] = [u, g]^k = 1.$$

Since $\tau(G') = 1$, clearly $[u, g] = 1$, for all $g \in G$, and so that $u \in Z(G)$. As $u \in A_1$ and $u \in Z(G)$, $Z(G) \cap A_1 \neq 0$. Observe that $Z(G) \cap A_1$ is a submodule of A_1. Since $Z(G) \cap A_1 \neq 0$, $Ass(Z(G) \cap A_1) \neq \emptyset$, by Lemma 3.2.1. Let $0 \neq a \in Z(G) \cap A_1$ be an element with maximal annihilator. Then $a^g = a$, for all $g \in G$, and $a^{g-1} = 1$, which implies that $g - 1 \in Ann(a)$. But since a is an element with maximal annihilator, $Ann(a)$ is a prime ideal, by Proposition 3.2.4. It follows that

$$Ann(a) \in Ass(Z(G) \cap A_1) \leq Ass(A_1).$$

Now since P is the only associated prime of A_1, $Ann(a) = P$, and so $g - 1 \in P$, for all $g \in G$. But the ideal generated by $g - 1$, for $1 \neq g \in G$, is the augmentation ideal I_G of R. So $I_G \subseteq R$. Now if I_G is properly contained in P, then R/P would be finite, but this is a contradiction, since $R/P \cong A_1$ is torsion free as Abelian group. Hence $I_G = P$. But $R/I_G = R/P \cong \mathbb{Z}$, and hence $A_1 \cong \mathbb{Z}$, which gives that

$$A_i/A_{i-1} \cong A_1 \cong \mathbb{Z},$$

for each $i = 1, 2, \cdots, n$, and so A is a finitely generated Abelian group. Then G/A and A are polycyclic groups and giving that G is a polycyclic group. Now since G is both metabelian and polycyclic, it follows that $Aut_{sn}(G) = 1$, by Lemma 5.1.1. This is a contradiction, since we have chosen that $1 \neq \alpha \in Aut_{sn}(G)$.

(ii) $Z(G) = 1$.

We may assume that G is a finitely generated non-polycyclic metabelian group, by Lemma 5.1.1.

Since $G \in \mathfrak{G} \cap \mathfrak{ANF}$, there exists a normal subgroup N of finite index in G such that N is residually finite-nilpotent, by Theorem 2.6.3. Now put $K = NA$, then K is residually finite-nilpotent, by Lemma 2.6.1. Since K has finite index in

G, it is a finitely generated metabelian group and $Aut_{sn}(K) \neq 1$. Observe that A is still finitely generated as $\mathbb{Z}[K/A]$-module, by Lemma 3.5.5.

Since $\tau(K) = 1$, $\omega(G) \leq \omega(K) = Z(K)$, by Lemma 2.6.2(ii), and so $K \leq C_G(\omega(G)$. Then K centralises u.

Since $u \in A_1$, $u^{g-1} = u$, for $g \in K$, and hence $I_K \subseteq Ann(u) \subseteq P$, giving $a^{g-1} = a$, for $a \in A_1$. Hence if $g_1, \cdots, g_t$ is a transversal for K in G and $a_1, \cdots, a_s$ generate A_1 as $\mathbb{Z}[G/A]$-module then $(a_i)^{g_j}$, $1 \leq i \leq s$, $1 \leq j \leq t$, generate A_1 as Abelian group. But

$$A_i/A_{i-1} \cong A_1,$$

for each $i = 1, 2, \cdots, n$, and so A is a finitely generated Abelian group. Then K/A and A are polycyclic groups and giving that K is a polycyclic group.

Now since G/K and K are polycyclic, G is polycyclic. This is a contradiction, since G is a non-polycyclic metabelian group.

This proves in any case that the group $Aut_{sn}(G) = 1$, and $\omega(G) = Z(G)$. ■

5.2 Proof of Theorem E, with non-trivial torsion radical

Before we begin a proof of Theorem E, for the non-trivial torsion radical case, we need the following two lemmas.

Lemma 5.2.1 *Let G be a finitely generated metabelian group with G' infinite and $\tau(G) \neq 1$. Then $[\omega(G), G] \leq \tau(G)$.*

Proof. We know that $\tau(G/\tau(G)) = 1$, by Lemma 2.3.4. We prove by considering when $G/\tau(G)$ is non-Abelian, and $G/\tau(G)$ is Abelian.

(i) Suppose that $G/\tau(G)$ is a non-Abelian group. Then $\omega(G/\tau(G)) = Z(G/\tau(G))$, by Lemma F, and so $\omega(G)\tau(G)/\tau(G) \leq \omega(G/\tau(G))$, and hence

$$[\omega(G), G] \leq \tau(G).$$

(ii) Suppose that $G/\tau(G)$ is an Abelian group. Then $G' \leq \tau(G)$, and so $[\omega(G)\tau(G)/\tau(G), G/\tau(G)] = 1$, and hence $[\omega(G), G] \leq \tau(G)$. In any case the result follows. ■

Lemma 5.2.2 *Let G be a finitely generated Abelian-by-nilpotent group with G' infinite and $\tau(G) \neq 1$ and set $\Gamma = Aut_{sn}(G)$. If $G/\tau(G)$ is Abelian, then*

$$[G, \Gamma] \leq \tau(G).$$

Proof. Since $\tau(G/\tau(G)) = 1$, by Lemma 2.3.4, $G/\tau(G)$ is torsion free Abelian. If $\alpha \in \Gamma$ acts as trivially on $G/\tau(G)$, then there is nothing to prove.

Suppose on the contrary that α acts as inversion on $G/\tau(G)$.

Now since $\omega(G)\tau(G)/\tau(G)$ is a normal subgroup of $G/\tau(G)$, α acts as inversion on $\omega(G)\tau(G)/\tau(G)$, that is

$$u^{\alpha}\tau(G) = u^{-1}\tau(G), \tag{5.1}$$

for $u \in \omega(G)$.

On other hand, since G' is infinite, α acts trivially on $\omega(G)$, by Theorem D(ii), and so α acts trivially on $\omega(G)\tau(G)/\tau(G)$, that is

$$u^{\alpha}\tau(G) = u\tau(G). \tag{5.2}$$

Hence from (5.1) and (5.2), it follows that $u\tau(G) = u^{-1}\tau(G)$, and so

$$u^2\tau(G) = \tau(G),$$

which implies that $u \in \tau(G)$, as $u\tau(G)$ has infinite order, and hence $\omega(G) \leq \tau(G)$.

But $[G, \alpha] \leq \omega(G) \leq \tau(G)$, and hence α acts trivially on $G/\tau(G)$. This is a contradiction, as α acts as inversion on $G/\tau(G)$ by our supposition. Hence the result follows. ■

The next result is a special case of Theorem E, and we prove this result as an intermediate step in the proof of Theorem E.

Lemma G *Let G be a finitely generated metabelian group. If $R(G) = 1$, then the group $Aut_{sn}(G) = 1$, and $\omega(G) = Z(G)$.*

Proof. Note that G' is infinite or $G' \leq R(G) = 1$ which implies G is Abelian, a contradiction, since G is non-Abelian. If $\tau(G) = 1$, then there is nothing to prove (by Lemma F). So we suppose the contrary. Since $R(G) = 1$, clearly

$$\tau(G) \cap Z(G) = 1,$$

and hence $Z(G)$ is a torsion free Abelian group.

We shall divide the proof into two cases, when $\omega(G)$ is a non-periodic Abelian group, and when $\omega(G)$ is a periodic Abelian group.

(i) Suppose that $\omega(G)$ is a non-periodic Abelian group.

We know that

$$[\omega(G), G] \leq \tau(G),$$

by Lemma 5.2.1.

On the other hand,

$$[\omega(G), G] \leq \omega(G).$$

Hence it follows that

$$[\omega(G), G] \leq \tau(G) \cap \omega(G) = T.$$

Now T is a torsion subgroup of $\omega(G)$ which is normal in G. Since $\omega(G)$ is finitely generated (see the remark after Theorem C of Chapter 4), T is finitely generated, and so finite. But since $R(G) = 1$, it follows that $T = 1$. Hence

$$[\omega(G), G] \leq \tau(G) \cap \omega(G) = 1,$$

which implies that $[\omega(G), G] = 1$, and so $\omega(G) \leq Z(G)$, giving

$$\omega(G) = Z(G)$$

and hence $\omega(G)$ is a torsion free Abelian group.

Let $\alpha \in Aut_{sn}(G)$. Then α acts trivially on $G/\omega(G)$, by Lemma 2.8.7, and also α acts trivially on $G/\tau(G)$, by Lemma F and Lemma 5.2.2, and so

$$g^{-1}g^{\alpha} \in \omega(G) \cap \tau(G) = 1,$$

for all $\alpha \in Aut_{sn}(G)$, and for all $g \in G$. Therefore the group $Aut_{sn}(G) = 1$, and $\omega(G) = Z(G)$, as required.

(ii) Suppose that $\omega(G)$ is a periodic Abelian group.

Then $\omega(G) \leq \tau(G)$. Since $\omega(G)$ is finitely generated (see the remark after Theorem C of Chapter 4), it follows that $\omega(G)$ is finite. Since $R(G) = 1$, clearly $\omega(G) = 1$, and hence $Aut_{sn}(G) = 1$, and $\omega(G) = Z(G) = 1$.

In any case $Aut_{sn}(G) = 1$, and $\omega(G) = Z(G)$.

This completes the proof of Lemma G. ∎

We are now in a position to complete the proof of Theorem E, part (a).

The proof of Theorem E, part (a). If $R(G) = 1$, then there is nothing to prove, (by Lemmas F and G), that is $Aut_{sn}(G) = 1$, and $\omega(G) = Z(G)$.

We suppose, on the contrary, that $R(G) \neq 1$. Then $G/R(G)$ has no non-trivial finite normal subgroup, in other words $R(G/R(G)) = 1$, and so

$$\tau(G/R(G)) \cap Z(G/R(G)) = 1,$$

and hence $Z(G/R(G))$ is a torsion free Abelian group.

Observe that since $R(G/R(G)) = 1$, it follows from Lemma G that

$$\omega(G/R(G)) = Z(G/R(G)).$$

Suppose that $\omega(G) \cap \tau(G) \neq \omega(G)$. Then $\omega(G)$ contains an element of infinite order. Since $\omega(G)R(G)/R(G) \leq Z(G/R(G))$ is torsion free,

$$\omega(G)R(G)/R(G) \cap \tau(G)/R(G) = 1,$$

and so $\omega(G)R(G) \cap \tau(G) = (\omega(G) \cap \tau(G))R(G) = R(G)$, and so

$$\omega(G) \cap \tau(G) \leq R(G).$$

We consider the cases

(a) $\omega(G) \cap \tau(G) = 1$ and
(b) $\omega(G) \cap \tau(G) \neq 1$ separately.

(*a*) Suppose that $\omega(G) \cap \tau(G) = 1$.

Since $\alpha \in Aut_{sn}(G)$ acts trivially on $G/\omega(G)$, by Lemma 2.8.7, and also α acts trivially on $G/\tau(G)$, by Lemma F and Lemma 5.2.2, $g^{-1}g^{\alpha} \in \omega(G) \cap \tau(G) = 1$, for all $g \in G$, which gives that $Aut_{sn}(G) = 1$. To see that $\omega(G) = Z(G)$, we know that

$$[\omega(G), G] \leq \tau(G),$$

by Lemma 5.2.1, and also

$$[\omega(G), G] \leq \omega(G),$$

it follows that

$$[\omega(G), G] \leq \tau(G) \cap \omega(G) = 1,$$

and hence $\omega(G) = Z(G)$.

(b) Suppose that $1 \neq T = \omega(G) \cap \tau(G) \leq R(G)$.

Then clearly T is the periodic subgroup of $\omega(G)$ and so finite. We know that $\alpha \in Aut_{sn}(G)$ acts trivially on $G/\omega(G)$, by Lemma 2.8.7, and also α acts trivially on $G/\tau(G)$, by Lemma F and Lemma 5.2.2, and so $g^{-1}g^{\alpha} \in \omega(G) \cap \tau(G) = T$, for all $\alpha \in Aut_{sn}(G)$ and for all $g \in G$. Now $g^{\alpha} \in gT$, and so $g^{\alpha} = gt$, for some $t \in T$. Since α acts trivially on $\omega(G)$, by Theorem D, if m is the order of T, we repeatedly apply α to g to get $g^{\alpha^m} = gt^m = g$, for every $\alpha \in Aut_{sn}(G)$. It follows that every element of $Aut_{sn}(G)$ is of finite order.

On the other hand, since $Aut_{sn}(G)$ stabilises the series $1 \leq T \leq G$ of length two, it follows that $Aut_{sn}(G)$ is Abelian, by Theorem 2.7.2. Hence $Aut_{sn}(G)$ is a periodic Abelian group.

Since $g^{-1}g^{\alpha} \in T \leq R(G)$, for all $g \in G$ and $\alpha \in Aut_{sn}(G)$, it follows that $[G, Aut_{sn}(G)] \leq T \leq R(G)$.

As G is finitely generated, let $g_1, g_2, \cdots, g_n$ be a set of generators for G. If α is any element of $Aut_{sn}(G)$, then $g_i^{\alpha} \in g_iT$ and so $g_i^{\alpha} \in \{g_it, t \in T\}$, a finite set for each $i = 1, 2, \cdots, n$. Since T is finite, it follows that $Aut_{sn}(G)$ has a finite set of elements, and hence $Aut_{sn}(G)$ is finite.

Next we show that $Z(G)$ has finite index in $\omega(G)$.

Since $\omega(G)$ is finitely generated (see the remark after Theorem C of Chapter 4) and $[\omega(G), G] \leq \tau(G) \cap \omega(G) = T$ is finite, $Z(G)$ has finite index in $\omega(G)$. The result follows. ■

We are now in a position to complete the proof of Theorem E, part (b).

The proof of Theorem E, part (b). Let G be a finitely generated metabelian group with finite derived subgroup G'. We note that when G' is finite

$$\tau(G) = R(G)$$

and G is centre-by-finite.

We aim to show that the group $Aut_{sn}(G)$ is a finite metabelian group.

Since G' is finite $\Gamma = Aut_{sn}(G)$ has a subgroup of index at most two centralising G/G', and we set

$$\Psi = C_{\Gamma}(G/G') = \{\alpha \in \Gamma : g^{\alpha}G' = gG', \ for\ all\ gG' \in G/G'\},$$

and so $[\Psi, F(G)] = 1$, by Theorem D(i).

We first show that Ψ is a finite Abelian group.

Ψ acts trivially on $G/\omega(G)$ and $\omega(G)$, by Lemma 2.8.7 and Theorem D(i) respectively, and so stabilises the series $1 \leq \omega(G) \leq G$, and hence Ψ is Abelian by Theorem 2.7.2.

Since G' is finite and $[G/G', \Psi] = 1$ and $[G/\omega(G), \Psi] = 1$, it follows that

$$[G, \Psi] \leq \omega(G) \cap G' = T,$$

where T is finite Abelian. By the above argument it follows that Ψ is a finite Abelian group.

Now if $\Gamma \neq \Psi$ and $\delta \in \Gamma \setminus \Psi$, then δ acts as inversion on non-periodic Abelian G/G' and hence has order two, and hence $\Gamma/\Psi \cong PAut(G/G')$, and so $\Gamma = \Psi \rtimes \langle \delta \rangle$ is a metabelian group. Since Γ/Ψ and Ψ is finite, Γ is finite. Therefore Γ is a finite metabelian group. This completes proof of Theorem E. ■

Corollary H *Let G be a finitely generated metabelian group. Assume that G' is torsion free as $\mathbb{Z}[G/G']$-module. Then the group $Aut_{sn}G = 1$, and*

$$\omega(G) = Z(G) = 1.$$

Proof. Let G be a finitely generated metabelian group. Let $\omega(G)$ be the Wielandt subgroup of G. Since G' is torsion free as $\mathbb{Z}[G/G']$-module, it is well known that G is residually nilpotent (see 15.3.9, [Rob82]).

Then by Lemma 2.6.2, $\omega(G) \leq Z_2(G)$. But since G' is torsion free as $\mathbb{Z}[G/G']$-module, by Lemma 3.5.4, $Z(G) = 1$, and so $\omega(G) \leq Z_2(G) = 1$. Thus

$$g^{-1}g^{\alpha} \in \omega(G) = 1,$$

for all $g \in G$, and for $\alpha \in Aut_{sn}(G)$. Therefore $Aut_{sn}(G) = 1$, and

$$\omega(G) = Z(G) = 1.$$

■

CHAPTER 6

6. EXTENSIONS BY MEANS OF AUTOMORPHISMS

In this chapter we consider whether the semidirect product of a finitely generated metabelian group G by $Aut_{sn}(G)$ is in general a finitely generated metabelian group.

In their paper [BCO92], Bryce, Cossey and Ormerod showed that if p is an odd prime and G is a non-abelian metabelian group of exponent p^2 with a power automorphism α of order p, then the semidirect product $G\langle\alpha\rangle$ is a metabelian group of exponent p^2 and the same class as G, where $\alpha \in \omega(G\langle\alpha\rangle)$.

We will give results which extend their results to a finitely generated metabelian group; and hence we prove that the semidirect product of a finitely generated metabelian group G by $\Gamma = Aut_{sn}(G)$ is a finitely generated metabelian group.

In more detail, we show that the Wielandt subgroup of the semidirect product $G \rtimes \Gamma$ is contained in $\omega(G)\Gamma$; the Fitting subgroup of the semidirect product $G \rtimes \Gamma$ is $F(G)\Gamma$; and the finite radical of the semidirect product $G \rtimes \Gamma$ is $R(G)\Gamma$.

In section 1 we give the main results of this chapter while section 2 presents the proofs of the results.

In section 3, we give examples. The first example shows that the complication given by centre-by-finite groups can not be avoided: it has non-Abelian $Aut_{sn}(G)$, $Aut_{sn}(G)$ does not centralise the Wielandt subgroup and $Aut_{sn}(G)$ contains outer automorphisms.

In the second example we show that the group $Aut_{sn}(G)$ need not in general be contained in $Inn(G)$, even in the case where $\omega(G)$ is centralised by $Aut_{sn}(G)$.

For the convenience of the reader, throughout this chapter we set $\Gamma = Aut_{sn}(G)$, and the semidirect product of G by Γ is Φ, that is $\Phi = G \rtimes \Gamma$.

6.1 The semidirect product of G by $Aut_{sn}(G)$

We have seen from Chapter 5 that the group $\Gamma = Aut_{sn}(G)$ of a finitely generated metabelian group G is a finite metabelian group (see Theorem E(b)) in genereal.

In this section we will state the main results of this chapter on the semidirect product of G by Γ and will give their proofs in the next section.

Our main results are the following.

Theorem I *Let G be a finitely generated metabelian group and $\Gamma = Aut_{sn}(G)$. Then $\Phi = G \rtimes \Gamma$, the semidirect product of G by Γ, is a finitely generated metabelian group, and $\omega(\Phi)$ is an Abelian group.*

This result easily yields the following Corollary, which gives more detailed information about the structure of Φ.

Corollary J *Let G be a finitely generated metabelian group. Let $\Phi = G \rtimes \Gamma$, the semidirect product of G by Γ. Then*

(i) $\omega(\Phi) \leq \omega(G)\Gamma$;

(ii) If G' is infinite, then $F(\Phi) = F(G)\Gamma$;

(iii) If G' is finite, then $F(\Phi) \leq F(G)\Gamma$; and

(iv) $R(\Phi) = R(G)\Gamma$.

We shall see in section 3 that the inclusion $\omega(G)\Gamma \leq \omega(\Phi)$ does not hold in general, and so we do not have $\omega(\Phi) = \omega(G)\Gamma$.

6.2 Proofs

Proof of Theorem I. Let G be a finitely generated metabelian group. To show that Φ is a metabelian group, it is sufficient to show that the derived subgroup of Φ is an Abelian group.

Observe that if the finite radical $R(G)$ of G is trivial ($R(G) = 1$), then $\Gamma = 1$, by Lemma G, and so $\Phi = G$, hence there is nothing to prove.

Hence suppose that $R(G) \neq 1$, and clearly we may also assume that $\Gamma \neq 1$.

As $\Phi = G \rtimes \Gamma$ the semidirect product of G by Γ and $\Gamma = \Psi \rtimes \langle \delta \rangle$ is metabelian, the derived subgroup of Φ is $\Phi' = G'[G, \Gamma]\Gamma'$. But by Lemma 2.8.7, $[G, \Gamma] \leq \omega(G)$. Observe that $\Gamma' = [\Psi, \langle \delta \rangle] \leq \Psi$. We know that Ψ is Abelian and centralises $F(G)$, that is $[\Psi, F(G) = 1]$. Since $G'\omega(G)$ is Abelian, by Lemma B, we have $[\Psi, G'\omega(G)] = 1$, and hence $G'\omega(G)\Gamma'$ is Abelian. Now it follows that

$$\Phi' = G'[G, \Gamma]\Gamma' \leq G'\omega(G)\Gamma',$$

is Abelian. Hence Φ' is an Abelian group, and therefore Φ is a metabelian group.

To see that Φ is finitely generated, clearly both G and Γ are finitely generated, and so $\Phi/G \cong \Gamma$ is finitely generated. Since G and Φ/G are finitely generated, it follows that Φ is finitely generated, by Lemma 2.2.3. Moreover, Φ is also infinite as G is infinite. Hence Φ is a finitely generated infinite metabelian group. Since Φ is a finitely generated infinite metabelian group, it follows, from Theorem A, that $\omega(\Phi)$ is an Abelian group.

This completes the proof of the Theorem. ■

Proof of Corollary J.

(i) Since Φ is a finitely generated metabelian group, by Theorem I, $\omega(\Phi)$ is an Abelian group.

Observe that every subnormal subgroup of G is subnormal in Φ, and so

$$\omega(\Phi) \cap G \leq \omega(G),$$

and hence $\omega(\Phi)\Gamma \cap \Phi \leq \omega(G)\Gamma$, which implies that $\omega(\Phi)\Gamma \leq \omega(G)\Gamma$, as required.

The result follows.

(ii) G' is infinite.

Clearly $F(\Phi) \cap G = F(G)$, by Lemma 2.4.2, and so

$$F(\Phi)\Gamma \cap \Phi = F(\Phi)\Gamma = F(G)\Gamma,$$

which implies that

$$F(\Phi) \leq F(G)\Gamma. \tag{6.1}$$

On the other hand, since Γ acts trivially on $F(G)$, that is $[F(G), \Gamma] = 1$, by Lemma D(ii), it follows that $F(G)\Gamma$ is nilpotent of the same class as $F(G)$, and so

$$F(G)\Gamma \leq F(\Phi). \tag{6.2}$$

Therefore from (6.1) and (6.2) we get $F(\Phi) = F(G)\Gamma$.

(iii) G' is finite.

Since $F(\Phi) \cap G = F(G)$, by Lemma 2.4.2,

$$F(\Phi)\Gamma \cap \Phi = F(\Phi)\Gamma = F(G)\Gamma,$$

and so

$$F(\Phi) \leq F(G)\Gamma.$$

The result follows.

(iv) Observe that $R(G) \leq R(\Phi)$, and so $R(\Phi) \cap G = R(G)$. Now

$$R(\Phi)\Gamma \cap \Phi = R(G)\Gamma,$$

which implies that

$$R(\Phi) \leq R(G)\Gamma. \tag{6.3}$$

On the other hand, since $R(\Phi)$ is the finite radical of Φ and $R(G)\Gamma$ is a finite normal subgroup of Φ, it follows that

$$R(G)\Gamma \leq R(\Phi). \tag{6.4}$$

Therefore, from (6.3) and (6.4), it follows that $R(\Phi) = R(G)\Gamma$. ■

6.3 Examples

The following example shows that the complication given by centre-by-finite groups can not be avoided: it has non-Abelian $Aut_{sn}(G)$, $Aut_{sn}(G)$ does not centralise the Wielandt subgroup and $Aut_{sn}(G)$ contains outer automorphisms.

Example 6.3.1 *Let $H = S_3 \times D_\infty$, where S_3 is the symmetric group of order six, that is $S_3 = \langle a, b : a^2 = b^3 = 1\rangle$, D_∞ is the infinite dihedral group, that is $D_\infty = \langle x, y : x^2 = 1,\ y^x = y^{-1}\rangle$. Note that $\langle y\rangle \cong \mathbb{Z}$. Set $G = S_3 \times \langle y\rangle$ and note $G \lhd H$. Let $\Gamma = Aut_{sn}(G)$. Then* *(i)* $\omega(G) = A_3 \times \langle y \rangle$, *and* $\omega(H) = A_3$*;*

(ii) $\Gamma = \langle \beta, \alpha\chi \rangle$*; where* $\beta = b$*,* $\alpha = a$ *and* $\chi = x$ *acting as automorphisms of* G *by conjugation in* H*.*

(iii) $\omega(G)\Gamma = \langle b, y \rangle \times \langle \beta, \alpha\chi \rangle \neq \omega(G\Gamma)$*.*

Proof. Let $G = S_3 \times \langle y \rangle$. We have $Z(S_3) = 1$ and S_3 has a normal subgroup of order three, A_3.

The derived subgroup of G is $G' = S_3' = A_3$, its centre is $Z(G) = \langle y \rangle$ and its torsion radical is $\tau(G) = R(G) = S_3$.

Observe that $Z(H) = Z(S_3) \times Z(D_\infty) = 1$, and so $Inn(H) \cong S_3 \times D_\infty = H$.

(i) We show that $\omega(G) = A_3 \times \langle y \rangle$. Observe that $\omega(S_3) = A_3$ and $A_3 \cong \mathbb{Z}_3$. If K *sn* G and $A_3 \nsubseteq K$, then $K \leq Z(G)$. Hence A_3 normalises every subnormal subgroup of G, and so $A_3 \leq \omega(G)$. It is known that $Z(G) \leq \omega(G)$, and hence

$$A_3 \times \langle y \rangle \leq \omega(G). \tag{6.5}$$

On the other hand, $\omega(G) \cap S_3 \leq \omega(S_3) = A_3$, and so $\omega(G) \cap S_3 \times \langle y \rangle \leq A_3 \times \langle y \rangle$, which implies that

$$\omega(G) \leq A_3 \times \langle y \rangle. \tag{6.6}$$

From (6.5) and (6.6), it follows that $\omega(G) = A_3 \times \langle y \rangle$.

Now we show that $\omega(H) = A_3$. We know that A_3 normalises every subnormal subgroup of H, and so $A_3 \leq \omega(H)$. On the other hand, $\omega(H) \cap S_3 \leq \omega(S_3) = A_3$, which implies that $\omega(H) \cap S_3 = A_3$. But $\omega(H) \cap D_\infty \leq \omega(D_\infty) = 1$, and so $\omega(H) \cap D_\infty = 1$. Hence it follows that $\omega(H) = A_3$.

(ii) We know that $\omega(G)/Z(G) \cong A_3 \neq 1$. Observe that

$$\Gamma \cap Inn(G) = \omega(G)/Z(G) \cong A_3,$$

which implies that $\Gamma \neq 1$, and $Inn(G) \cong S_3$. Clearly $A_3 = \langle b \rangle \leq \Gamma$ as inner automorphisms.

Next we show that $ax \in \Gamma$. Observe that ax acts as a power automorphism on $\langle y \rangle$, that is $y^{ax} = y^x = y^{-1}$.

Moreover ax also acts as a power automorphism on $A_3 = \langle b \rangle$, that is

$$b^{ax} = (b^a)^x = (b^{-1})^x = b^{-1}.$$

To see that $ax \in \Gamma$, let L be a subnormal subgroup of G. Let $by^i \in L$, for some $i \in \mathbb{Z}$. Then $(by^i)^{ax} = b^{ax}(y^i)^{ax} = b^{-1}y^{-i} \in L$, and so ax normalises every subnormal subgroup of G, and hence $ax \in \Gamma$.

Now we can see that $\Gamma = \langle \beta, \alpha\chi \rangle$, as $\beta = b$ is inner automorphism.

(iii) To show that $\omega(G)\Gamma = \langle b, y \rangle \times \langle \beta, \alpha\chi \rangle \neq \omega(G\Gamma)$.

Observe that $\omega(G\Gamma) \cong \omega(H) = A_3$, by (i) above, and hence $\omega(G)\Gamma = \langle b, y \rangle \times \langle \beta, \alpha\chi \rangle \neq \omega(G\Gamma) = A_3$. The result follows. ■

The next example shows that the group $Aut_{sn}(G)$, of a finitely generated metabelian group G, need not in general be contained in $Inn(G)$, even in the case where $\omega(G)$ is centralised by $Aut_{sn}(G)$.

Recall that a group G is a *semidirect product* of a group L by a group Q, denoted by $G = L \rtimes Q$, if L is a normal subgroup of G and $G/L \cong Q$ such that $G = LQ$, and $L \cap Q = 1$. Q is called a *complement* for L in G.

We remind the reader of the standard wreath product here. The standard wreath product $W = H \wr K$ of two groups H and K, is the semidirect product of $H^{(K)}$ by K, that is,

$$W = H^{(K)} \rtimes K = H^{(K)}K; \;\; H^{(K)} \lhd W, \;\; H^{(K)} \cap K = 1,$$

where $H^{(K)}$ is the direct product of isomorphic copies of H indexed by elements of K, that is,

$$B = H^{(K)} = \prod_{k \in K} H_k.$$

The normal subgroup B of W is called the *base group*. Let $b \in B$ with $b_k =$ the k-component of b. If $x \in K$, we define b^x by the rule $(b^x)_k = b_{kx^{-1}}$, where $k \in K$. K permutes the direct factors H_k according to the right regular multiplication of K, that is $(H_k)^x = H_{kx}$. Hence K is a group of automorphisms of B.

Example 6.3.2 *Let $W = \mathbb{Z} \wr \mathbb{Z}_p$, the standard wreath product of $\mathbb{Z}$ by $\mathbb{Z}_p$. Let $B = \mathbb{Z}^{(\mathbb{Z}_p)}$ be the base group of W. Set $M = [B, \mathbb{Z}_p]$, (clearly M is the augmentation ideal of B, is normalised by $\mathbb{Z}_p$). Let*

$$H = \langle x, y : x^{p^2} = y^p = 1, x^y = x^{1+p} \rangle,$$

where p is an odd prime.

Define the action of H on M as follows: x acts on M as t, where $t \in \mathbb{Z}_p$, that is, $m^x = m^t$; and y centralises M. Now set $K = M\langle x\rangle$ and $G = MH$, and $K \lhd G$. Then $Aut_{sn}(K) \nsubseteq Inn(K)$.

Proof. Clearly M is free Abelian. Observe that M is a subgroup of the base group B which is normal in G.

Since $x^y = x^{1+p}$, the element y induces an automorphism on $\langle x\rangle$ whose order is p, and so $y \notin Z(G)$; but y centralises M. Let $m \in M$.

A routine calculation gives us the following.

(i) $(xm)^p = x^p$.

Next we show that the element y is in $\omega(G)$. To see that we need to show the following.

(ii) The element y normalises every subnormal subgroup of G.

Let L be a subnormal subgroup of G. If $L \leq M$, then $L^y = L$, there is nothing to prove, as y centralises M.

Suppose that $L \nsubseteq M$. Let $xm \in L$, where $x \in H$ and $m \in M$. Then

$$(xm)^y = x^y m^y = x^{1+p}m = xx^p m = (xm)(x^p) = xm(xm)^p = (xm)^{1+p},$$

as $(xm)^p = x^p$ by (i) above. Hence $(xm)^y = (xm)^{1+p}$. It follows that $L^y = L$. This proves that the element y normalises every subnormal subgroup of G, which implies that $y \in \omega(G)$, but $y \notin Z(G)$.

We know that the torsion radical $\tau(G) = \langle x^p, y\rangle = R(G)$, the finite radical of G. Next we show that $m^i \notin Z(G)$, for some $i \in \mathbb{Z}$. Suppose that $m^i \in Z(G)$. Then $(m^i)^x = m^i$, for all $x \in H$. But $(m^i)^x = (m^x)^i = (m[m,x])^i = m^i[m,x]^i = m^i$, and so $[m,x]^i = 1$. Since $\tau(M) = 1$, clearly $[m,x] = 1$. This is a contradiction, since $[m,x] \neq 1$. Hence $m^i \notin Z(G)$. It follows easily that $\langle x^p\rangle = Z(G)$. Observe also that $\tau(K) = \langle x^p\rangle = R(K)$, and so $Z(K) = \tau(K) = \langle x^p\rangle = R(K)$, and hence

$$Z(K) = Z(G) = \langle x^p\rangle.$$

Clearly $\langle x^p\rangle = Z(G) \leq \omega(G)$, and $y \in \omega(G)$, and so

$$\tau(G) = \langle x^p, y\rangle \leqq \omega(G).$$

Now since $\tau(G)/Z(G) \lhd \omega(G)/Z(G)$ is finite, it follows that $\omega(G)/\tau(G)$ is also finite, but $\omega(G)/\tau(G) \lhd G/\tau(G)$ and $\tau(G/\tau(G)) = 1$, which implies that

$$\omega(G) = \tau(G).$$

Thus

$$Z(G) \leq \omega(G) = \tau(G) = \langle x^p, y \rangle.$$

Since $y \notin K$, clearly $y \notin \omega(K)$. Observe that $\tau(K) = Z(K) \leq \omega(K)$. We know that $\omega(K)/\tau(K) \lhd K/\tau(K)$ and $\tau(\omega(K)/\tau(K)) = 1$. But since $Z(K) = \tau(K)$, $\omega(K)/\tau(K)$ is finite, and so

$$\omega(K) = Z(K) = \tau(K).$$

Hence it follows that $Inn(K) \cap Aut_{sn}(K) = \omega(K)/Z(K) = 1$.

Note that $y \notin \omega(K) = Z(K) = Z(G)$, but y induces an automorphism on $\langle x \rangle$ whose order is p, and also y centralises M, and hence y normalises every subnormal subgroup of K, which implies that $y \in Aut_{sn}(K)$.

Let α be any element of $Aut_{sn}(K)$. Since $k^{-1}k^{\alpha} \in \tau(K) \cap \omega(K) = \omega(K) = \langle x^p \rangle$ and $k^{\alpha^p} = k$, for all $\alpha \in Aut_{sn}(K)$ and for all $k \in K$. Hence α has exponent p. It follows easily that $Aut_{sn}(K) \cong \mathbb{Z}_p$, and $Inn(K) \cap Aut_{sn}(K) = 1$. Hence this shows that $Aut_{sn}(K) \nsubseteq Inn(K)$. ■

CHAPTER 7

7. ABELIAN-BY-NILPOTENT GROUPS

Recall from Chapter 5 our main result (Theorem E), that if G is a finitely generated infinite metabelian group, then the group $Aut_{sn}(G)$ is a finite Abelian group when G' is infinite, and a finite metabelian group when G' is finite. Moreover $[G, Aut_{sn}(G)] \leq R(G)$, and $Z(G)$ has finite index in $\omega(G)$ if G' is infinite.

Now we ask whether these results can be extended to a finitely generated infinite Abelian-by-nilpotent group. As we have seen in Chapter 3, the connection of commutative algebra with a finitely generated metabelian group plays a crucial role in the proof of our main result Theorem E for the trivial torsion radical ($\tau(G) = 1$) case, (see proof of Lemma F). It is an important step specially in the proof of Theorem E (*iii*) to show that $Z(G)$ has finite index in $\omega(G)$, more precisely $\omega(G) = Z(G)$ when $\tau(G) = 1$, and $Aut_{sn}(G) = 1$. The reason for the connection with commutative algebra is that there exists an Abelian normal subgroup A of a finitely generated metabelian group G such that G/A is finitely generated Abelian and hence A is a finitely generated as $R = \mathbb{Z}[G/A]$-module, where $R = \mathbb{Z}[G/A]$ is the integral group ring of G/A. For the metabelian case this commutative algebra is sufficient to give us the information we need. But in a finitely generated Abelian-by-nilpotent group the situation is different. For instance, if G is a finitely generated Abelian-by-nilpotent group which is non-metabelian, then we can not find an Abelian normal subgroup B of G such that G/B is Abelian. Because of this limitation we can not use the techniques and results from commutative algebra for the finitely generated Abelian-by-nilpotent case. As a consequence of this the following results are obtained.

Theorem K *Let G be a finitely generated Abelian-by-nilpotent group.*

(i) If G' is infinite, then the group $Aut_{sn}(G)$ is a finitely generated Abelian group.

(ii) If G' is finite, then the group $Aut_{sn}(G)$ is a finite metabelian group.

This result easily gives the following Corollaries.

Corollary L *Let G be a finitely generated Abelian-by-nilpotent group with G' infinite and set $\Gamma = Aut_{sn}(G)$.*

If $\omega(G) = Z(G)$ then Γ is a finite Abelian group, and

$$[G, \Gamma] \leq R(G).$$

In particular, if $R(G) = 1$, then $\Gamma = 1$.

Corollary M *Let G be a finitely generated Abelian-by-nilpotent group and let $H = C_G(\omega(G))$, and set $\Gamma = Aut_{sn}(G)$.*

If H' is infinite, then Γ is a finite Abelian group, and

$$[\Gamma, G] \leq \omega(G) \leq Z(H) \leq R(G).$$

In particular, if $R(G) = 1$, then $\Gamma = 1$.

We give results which generalise the metabelian case (Theorem I) of Chapter 6 to the class of finitely generated Abelian-by-nilpotent groups.

Our main results in this case are the following.

Theorem N *Let G be a finitely generated Abelian-by-nilpotent group and $\Gamma = Aut_{sn}(G) = \Psi \rtimes \langle\delta\rangle$, where*

$$\Psi = C_\Gamma(G/G') = \{\alpha \in \Gamma : g^\alpha G' = gG',\ for\ all\ gG' \in G/G'\}$$

and $\delta \in \Gamma \setminus \Psi$. Let A be a maximal Abelian normal subgroup of G such that G/A is nilpotent, and set $B = A\Psi$.

(i) *The semidirect product of G by Γ, $\Phi = G \rtimes \Gamma$, is a finitely generated Abelian-by-nilpotent group.*

(ii) *Φ/B and G/A have the same nilpotency class.*

(iii) *$\omega(\Phi)$ is an Abelian group.*

This result gives the following Corollary (compare with Corollary J of Chapter 6).

Corollary O *Let G be a finitely generated Abelian-by-nilpotent group and let $\Phi = G \rtimes \Gamma$, the semidirect product of G by Γ.*

(i) $\omega(\Phi) \leq \omega(\Phi)\Gamma$;

(ii) If G' *is infinite,* $F(\Phi) = F(G)\Gamma$; *and*

(iii) If G' *is finite* $F(\Phi) \leq F(G)\Gamma$; *and*

(iv) If Γ *is finite, then* $R(\Phi) = R(G)\Gamma$.

In the next section we present the proofs of all the results in this chapter.

7.1 Proofs

Proof of Theorem K. Let G be a finitely generated Abelian-by-nilpotent group and $\Gamma = Aut_{sn}(G)$.

(i) G' is infinite.

We know that $\omega(G)$ is an Abelian group, by Theorem A, and so $\omega(G) \leq F(G)$. Γ centralises $F(G)$, by Lemma D(ii), and so $[\Gamma, \omega(G)] = 1$.

On the other hand, $[\Gamma, G/\omega(G)] = 1$, by Lemma 2.8.7, and so Γ stabilises the series $1 \leq \omega(G) \leq G$ of length two, which implies that Γ is an Abelian group, by Theorem 2.7.2.

We note that $\omega(G)$ is finitely generated by the remark after Theorem C of Chapter 4.

Let $g_1, g_2, \cdots, g_n$ be a set of generators for G.

Let $\alpha \in \Gamma$, and $g_i \in G$. Consider the mapping $\varphi_i : \alpha \longrightarrow [g_i, \alpha]$. Then φ_i is a homomorphism from Γ into $\omega(G)$. To see that φ_i is a homomorphism, let $\beta \in \Gamma$. Then

$$\varphi_i(\alpha\beta) = [g_i, \alpha\beta] = [g_i, \alpha][g_i, \beta] = \varphi_i(\alpha)\varphi_i(\beta),$$

and hence φ_i is a homomorphism.

Observe that

$$\prod_{i=1}^{n} \varphi_i(\alpha) = \prod_{i=1}^{n} [g_i, \alpha] \in \omega(G)^n.$$

Now consider the map $\phi : \alpha \longrightarrow \prod_{i=1}^{n} \varphi_i(\alpha)$. Clearly ϕ is a homomorphism from Γ into $\omega(G)^n$. Let Im $\phi = \phi(\Gamma)$, the image of ϕ, and denote the kernel of ϕ by Ker ϕ . Observe that Ker $\phi = 1$, and so ϕ is a monomorphism. We know that Im ϕ is contained in $\omega(G)^n$, that is

$$Im\ \phi \leq \omega(G)^n.$$

Since Ker $\phi = 1$, it follows that $\Gamma \cong Im\ \phi \leq \omega(G)^n$. We know that $\omega(G)$ is a finitely generated Abelian group, and so $\omega(G)^n$ is also finitely generated, and hence Γ is finitely generated.

(ii) G' is finite.

The proof of (ii) just follows from the proof of Theorem E, part (b).

This completes the proof of the Theorem. ■

Proof of Corollary L. Let G be a finitely generated Abelian-by-nilpotent group with G' infinite and set $\Gamma = Aut_{sn}(G)$.

Since this result is true for finitely generated metabelian groups we may assume that G is not metabelian and hence $G/\gamma_3(G)$ is non-Abelian and nilpotent.

Since $G \in \mathfrak{G} \cap \mathfrak{A}\mathfrak{N}$, there exists a positive integer n such that

$$\gamma_n(G) \cap Z(G) = 1,$$

by 15.3.6 of [Rob82].

We can choose $n \geq 3$ so that $G/\gamma_n(G)$ is non-Abelian and nilpotent.

Now we consider when $G/\gamma_n(G)$ is non-periodic nilpotent, and when $G/\gamma_n(G)$ is periodic nilpotent.

(i) Suppose that $G/\gamma_n(G)$ is a non-periodic nilpotent group.

Since $G/\gamma_n(G)$ is not Abelian, then $G/\gamma_n(G)$ has no non-trivial power automorphisms, by Lemma 2.5.3(ii), and so Γ centralizes $G/\gamma_n(G)$, and hence

$$[G/\gamma_n(G), \Gamma] = 1,$$

and so

$$[\Gamma, G] \leq \gamma_n(G). \tag{7.1}$$

On the other hand,

$$[\Gamma, G] \leq Z(G), \tag{7.2}$$

by Lemma 2.8.7.

Now from (7.1) and (7.2), it follows that

$$[\Gamma, G] \leq \gamma_n(G) \cap Z(G) = 1,$$

and hence $\Gamma = 1$.

(ii) Suppose that $G/\gamma_n(G)$ is a periodic nilpotent group.

Since $G/\gamma_n(G)$ is finitely generated and periodic nilpotent, $G/\gamma_n(G)$ is finite, by 5.4.11 of [Rob82]. As $\gamma_n(G) \cap Z(G) = 1$ and $Z(G) \cong \gamma_n(G)Z(G)/\gamma_n(G)$, this implies that $Z(G)$ is finite, and hence $Z(G) \leq R(G)$. Since by hypothesis $\omega(G) = Z(G)$, it follows that $[\Gamma, G] \leq Z(G) \leq R(G)$, by Lemma 2.8.7. Hence the result follows. ∎

Proof of Corollary M. Let G be a finitely generated Abelian-by-nilpotent group and set $\Gamma = Aut_{sn}(G)$. Let $H = C_G(\omega(G))$. Then H has finite index in G, by Theorem C. Observe that $\omega(G) \leq Z(H)$, by definition of H, and assume H' infinite.

Since this result is true for finitely generated metabelian groups we may assume that H is not metabelian and hence $H/\gamma_3(H)$ is non-Abelian and nilpotent.

Since $G \in \mathfrak{G} \cap \mathfrak{A}\mathfrak{N}$, there exists a positive integer n such that

$$\gamma_n(H) \cap Z(H) = 1,$$

by 15.3.6 of [Rob82].

Again we can choose $n \geq 3$ so that $H/\gamma_n(H)$ is non-Abelian and nilpotent.

Now we consider when $H/\gamma_n(H)$ is non-periodic nilpotent, and when $H/\gamma_n(H)$ is periodic nilpotent.

(i) Suppose that $H/\gamma_n(H)$ is a non-periodic nilpotent group.

Since $H/\gamma_n(H)$ is not Abelian, $H/\gamma_n(H)$ has no non-trivial power automorphisms, by Lemma 2.5.3(ii), and so Γ centralizes $H/\gamma_n(H)$, and hence

$$[H/\gamma_n(H), \Gamma] = 1,$$

and so

$$[\Gamma, H] \leq \gamma_n(H). \tag{7.3}$$

On the other hand,

$$[\Gamma, H] \leq Z(H), \tag{7.4}$$

by Lemma 2.8.7.

Now from (7.3) and (7.4), it follows that

$$[\Gamma, H] \leq \gamma_n(H) \cap Z(H) = 1,$$

and hence $[\Gamma, H] = 1$.

(ii) Suppose that $H/\gamma_n(H)$ is a periodic nilpotent group.

Since $H/\gamma_n(H)$ is finitely generated periodic nilpotent, it is finite by 5.4.11 of [Rob82]. As $\gamma_n(H) \cap Z(H) = 1$ and $Z(H) \cong \gamma_n(H)Z(H)/\gamma_n(H)$, implies that $Z(H)$ is finite and hence $Z(H) \leq R(H)$. Thus $[\Gamma, H] \leq Z(H) \leq R(H)$, by Lemma 2.8.7. Since $\omega(G) \leq Z(H)$, by definition of H, it follows that $[\Gamma, G] \leq \omega(G) \leq Z(H) \leq R(G)$, and hence if $R(G) = 1$ then $\Gamma = 1$.

Hence the result follows. ∎

Proof of Theorem N. Let G be a finitely generated Abelian-by-nilpotent group and A be a maximal Abelian normal subgroup of G such that G/A is a nilpotent group. Recall that $\omega(G) \leq A$.

Let $\Gamma = Aut_{sn}(G) = \Psi \rtimes \langle\delta\rangle$, and let $\Phi = G \rtimes \Gamma$ the semidirect product of G by Γ.

Observe that Ψ is a finitely generated Abelian group, by Theorem K(i), and $[\Psi, F(G)] = 1$, by Theorem D(ii), and hence $[\Psi, A] = 1$. Since both Ψ and A are Abelian groups and $[\Psi, A] = 1$, $B = A\Psi$ is an Abelian normal subgroup of Φ.

(i) To show that $\Phi = G \rtimes \Gamma$ is an Abelian-by-nilpotent group, it is sufficient to show that Φ/B is a nilpotent group.

Clearly

$$\Phi/B \cong G/A \times \langle\delta\rangle, \tag{7.5}$$

where $\delta \in \Gamma \setminus \Psi$. Since G/A is nilpotent and $\langle\delta\rangle$ has order at most 2, it follows that Φ/B is nilpotent.

To see that Φ is finitely generated, clearly both G and Γ are finitely generated, and so $\Phi/G \cong \Gamma$ is finitely generated. Since G and Φ/G are finitely generated, it follows that Φ is finitely generated, by Lemma 2.2.3. Moreover, Φ is also infinite as G is infinite. Hence Φ is a finitely generated infinite Abelian-by-nilpotent group.

(ii) That Φ/B and G/A have the same nilpotency class follows from (7.5) above.

(iii) Since Φ is a finitely generated infinite Abelian-by-nilpotent group (see (i) above), it follows that $\omega(\Phi)$ is an Abelian group, by Theorem A.

This completes the proof of the Theorem. ∎

Proof of Corollary O. Let G be a finitely generated Abelian-by-nilpotent group and $\Gamma = Aut_{sn}(G)$. Let $\Phi = G \rtimes \Gamma$, the semidirect product of G by Γ.

(i) Since Φ is a finitely generated Abelian-by-nilpotent group, $\omega(\Phi)$ is an Abelian group, by Theorem A.

Observe that every subnormal subgroup of G is subnormal in Φ, and so

$$\omega(\Phi) \cap G \leq \omega(G),$$

and hence $\omega(\Phi)\Gamma \cap \Phi \leq \omega(G)\Gamma$, which implies that $\omega(\Phi)\Gamma \leq \omega(G)\Gamma$. Hence it follows that $\omega(\Phi) \leq \omega(G)\Gamma$.

(ii) G' is infinite.

Clearly $F(\Phi) \cap G = F(G)$, by Lemma 2.4.2, and so

$$F(\Phi)\Gamma \cap \Phi = F(\Phi)\Gamma = F(G)\Gamma,$$

which implies that

$$F(\Phi) \leq F(G)\Gamma. \tag{7.6}$$

On the other hand, since Γ acts trivially on $F(G)$, that is $[F(G), \Gamma] = 1$, by Lemma D(ii), it follows that $F(G)\Gamma$ is nilpotent of the same class as $F(G)$, and so

$$F(G)\Gamma \leq F(\Phi). \tag{7.7}$$

Therefore from (7.6) and (7.7) we get $F(\Phi) = F(G)\Gamma$.

(iii) G' is finite.

Since $F(\Phi) \cap G = F(G)$, by Lemma 2.4.2,

$$F(\Phi)\Gamma \cap \Phi = F(\Phi)\Gamma = F(G)\Gamma,$$

and so

$$F(\Phi) \leq F(G)\Gamma.$$

The result follows.

(iv) Observe that $R(G) \leq R(\Phi)$, and so $R(\Phi) \cap G = R(G)$. Now

$$R(\Phi)\Gamma \cap \Phi = R(G)\Gamma$$

and then $R(\Phi)\Gamma \cap \Phi = R(\Phi)\Gamma = R(G)\Gamma$, which implies that

$$R(\Phi) \leq R(G)\Gamma. \tag{7.8}$$

On the other hand, since $R(\Phi)$ is the finite radical of Φ and Γ is also finite, $R(G)\Gamma$ is a finite normal subgroup of Φ, it follows that

$$R(G)\Gamma \leq R(\Phi). \tag{7.9}$$

Therefore, from (7.8) and (7.9), it follows that $R(\Phi) = R(G)\Gamma$. ∎

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